

SOME QUASI-EVERYWHERE RESULTS ON WIENER SPACE

Mathew David Penrose

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University of Edinburgh**

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ABSTRACT

The main part of this thesis is concerned with proving that certain properties of a Brownian path hold quasi-everywhere with respect to the Ornstein-Uhlenbeck process in Wiener space, a diffusion in the space of paths in Euclidean space with initial and stationary measure given by Wiener measure. A property of a Brownian path is said to hold quasi-everywhere if the Ornstein-Uhlenbeck process in Wiener space never visits the set of paths failing to have that property.

We prove that in three-dimensional space, quasi-every Brownian path has a "self-intersection local time" in the sense of Rosen. Using this local time we show that for quasi-every path the set of self-intersections of the path has a Hausdorff dimension of one, generalising a result of Fristedt from an "almost everywhere" to a "quasi-everywhere" statement. We generalise the analogous result in two dimensions (due to Taylor) similarly.

Using related local time methods, we show that for quasi-every path on the line the inverse image of the origin has Hausdorff dimension of one half.

The last chapter of the thesis is a study of the stochastic process given by taking the minimum of a large family of independent Bessel processes, pointwise over time. We find the limiting distribution (in the space of continuous paths) of an appropriate re-normalisation (in space and time) of this process as the size of

the family tends to infinity. The limit process is given by taking the pointwise minimum modulus of a family of paths in Euclidean space, with initial positions given by a homogeneous Poisson process and subsequent evolution given by independent Brownian paths.

We also discuss the appropriate re-normalisations (in space only) of the maximal and minimal functions of the process given by a pointwise minimum of Bessel processes.

The material contained in this work is original, except where explicitly mentioned to the contrary.

This thesis has been composed by myself.

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INTRODUCTION

The properties of a Brownian path $b(t)$ in d -dimensional Euclidean space have been extensively studied. Here are three examples of results obtained. Firstly, if $d \leq 3$, then the path intersects itself (Dvoretzky, Erdős and Kakutani (1950)). Secondly, if $d=1$ the Hausdorff dimension of the set of times t for which $b(t) = 0$ is $\frac{1}{2}$, (Taylor (1955)). Thirdly (Rosen (1983)), if $d=3$ then there exists a useful concept of a "self-intersection local time" α for the path $b(t)$, given formally by

$$\alpha = \iint_B \delta(b(s)-b(t)) ds dt ,$$

for any Borel set B in $[0,1] \times [0,1]$ which is strictly separated from the diagonal $\{(s,t): s=t\}$ (Here δ is the Dirac delta function).

All these results refer to all Brownian paths $b(\cdot)$ in \mathbb{R}^d except for those paths lying in some exceptional set with zero Wiener measure. Recently, however, there has been much interest in considering different classes of exceptional set. In particular, a subset of Wiener space which is never visited by an Ornstein-Uhlenbeck process in Wiener space with initial position distributed by Wiener measure is said to have zero capacity (with respect to the Ornstein-Uhlenbeck process). The class of such sets is a proper subset of the class of sets with zero Wiener measure. Many classical properties of Brownian motion hold "quasi-everywhere", that is for all paths outside a set of zero capacity. For example, since an Ornstein-Uhlenbeck process in Wiener space can be derived by re-normalising a two-parameter Wiener process (or "Brownian sheet"),

the results on the quadratic variation and the law of the iterated logarithm for quasi-every path follow from results on the Brownian sheet by Williams (1982) and Walsh (1982) respectively.

An equivalent way to describe the sets of zero capacity in Wiener space is to use the Dirichlet form on Wiener space associated with the Ornstein-Uhlenbeck operator, and to consider the associated capacity, as in Fukushima (1984). Using this potential-theoretic method Fukushima obtained many quasi-everywhere results.

Some results have been shown not to hold for quasi-every path, although they hold for almost every path with respect to Wiener measure. For example, Lyons (1986) showed by potential-theoretic methods that if d is either 4 or 5, the set of self-intersecting paths in \mathbb{R}^d has non-zero capacity, although it has zero Wiener measure. Mountford (1988) re-derived Lyons's results using local time methods on the Brownian sheet, related to methods used here.

In this thesis, we shall show that each of the three properties of a Brownian path which we described at the start holds for quasi-every path $b(t)$. We shall do this by using properties of the Brownian sheet rather than by potential-theoretic methods, analysing the self-intersections of the Brownian path via the self-intersection local time.

Apart from the theoretical interest in generalising results on Brownian motion from "almost everywhere" to "quasi-everywhere", the study of Brownian self-intersections is motivated by questions concerning the statistical properties of long polymer chains. As explained in Freed (1981), it is desirable to model such chains as

self-avoiding "Brownian" paths in space, so that a measure on the set of self-avoiding paths in \mathbb{R}^3 is required. It is possible that for some potential theory on Wiener space, the set of self-avoiding paths in \mathbb{R}^3 has positive capacity. (Such a potential theory might be provided by one of the (r,p) capacities investigated by Takeda (1984).) If so, then the equilibrium measure of the set of self-avoiding paths with respect to such a potential theory might provide a suitable polymer measure. The results here show that this approach fails in the case of the potential theory associated with the Ornstein-Uhlenbeck process in Wiener space.

We now outline our results in greater detail. Let τ be the time parameter of an Ornstein-Uhlenbeck process in Wiener space, so that $b_\tau(\cdot)$ is a Brownian path for each τ and varies continuously in τ . The self-intersection local time (cf. Rosen (1983)) of the path $b_\tau(t)$, relative to a set B in the upper triangle of \mathbb{R}_+^2 , is given formally by

$$\varphi(\tau, B) = \int_B \delta(b_\tau(t) - b_\tau(s)) \, ds dt.$$

By Fourier analysis, this local time can be expressed as an integral over \mathbb{R}^d of a d -parameter stochastic process. By applying Kolmogorov's lemma, we show in Chapter 3 that the local time is continuous in the time-parameter τ of the Ornstein-Uhlenbeck process in Wiener space. Hence quasi-every path has an intersection local time (Komatsu and Takashima (1984) prove the quasi-everywhere existence in the case $d=2$ of an "intersectional local time" defined somewhat differently from ours).

In Chapter 5 we examine the probability distribution of the random variable α given by the intersection local time of two independent Brownian motions b and b' in \mathbb{R}^3 running for unit time, given formally by

$$\alpha = \int_0^1 \int_0^1 \delta(b'(t) - b(s)) \, ds dt .$$

We make estimates on the probabilities that α is very small, finding a sequence $\epsilon_n \rightarrow 0$ such that

$$P[\alpha < \epsilon_n] < c_1 \exp(c_2(\log \epsilon_n)/\log|\log \epsilon_n|).$$

We then consider (Chapter 6) a subset B_0 of the unit square which is a union of countably many squares (of geometrically decreasing size) touching the diagonal. We show that there is so little of B_0 touching the diagonal that the self-intersection local time $\varphi(\tau, B_0)$ is continuous in τ . On the other hand, we use the estimates in Chapter 5, the Borel-Cantelli lemma and the Hölder continuity of φ in τ to show that there is so much of B_0 touching the diagonal that $\varphi(\tau, B_0)$ is strictly positive for all τ , almost surely.

This gives us the existence of self-intersections in the unit time-interval, for quasi-every Brownian path; in fact, we show in Chapter 7 that the Hausdorff dimension of the set of self-intersections is 1, for quasi-every path. This is a generalisation to "quasi-everywhere" of an "almost everywhere" result of Fristedt (1967).

The proof that quasi-every path $b(t)$ in \mathbb{R} has a level set $b^{-1}\{0\}$ of Hausdorff dimension $\frac{1}{2}$ follows fairly similar lines

(although a different argument is needed to show that the local time is strictly positive quasi-everywhere).

Finally, in Chapter 8 we investigate the asymptotics, as $n \rightarrow \infty$, of the process

$$M_n(t) = \min \{R_i(t), 1 \leq i \leq n\},$$

where $R_i(t)$, $i=1,2,3,\dots$ are independent Bessel processes of index d . In the introduction to that chapter, the relation to the work in the rest of the thesis is discussed, along with some physical problems (especially those of rates of chemical reaction in a suspension fluid) related to the study of $M_n(\cdot)$. The case where the R_i 's are Brownian motions (rather than Bessel processes) was considered by Brown and Resnick (1977). We show that the critical re-normalisation of the time-parameter of M_n is by a factor of $n^{-2/d}$, while the correct re-normalisation of the random variable $M_n(t)$ is by a factor of $n^{1/d}$. We obtain results on the narrow ("weak") convergence in $C[0,\infty)$ of the re-normalised processes.

In addition we consider the maximal and minimal functions of $M_n(\cdot)$ given by

$$M_n^*(s,t) = \max_{s \leq \tau \leq t} M_n(\tau), \quad (M_n)_*(s,t) = \min_{s \leq \tau \leq t} M_n(\tau).$$

We show that the processes $(n/\log n)^{1/d} M_n^*(0, \cdot)$ converge narrowly in $C[0,\infty)$ to a deterministic process, while for $d > 2$ and $0 < s < t$, $n^{-1/(d-2)} (M_n)_*(s,t)$ converges weakly to an exponential-type distribution. Thus we can compare the rates at which the distributions of $M_n(t)$, $M_n^*(s,t)$ and $(M_n)_*(s,t)$ decrease to zero as n tends to infinity.

With the exception of the results on quasi-everywhere properties of the local time of Brownian motion on the line, the results in this thesis have been submitted for publication in Penrose (1988a), (1988b), (1988c).

1.1 The Ornstein-Uhlenbeck process on the line. An

Ornstein-Uhlenbeck process $(X(\tau), \tau \geq 0)$ on the line, with initial distribution given by a standard normal, can be obtained by re-normalising Brownian motion $b(t)$ on the line to the process

$$X(\tau) = e^{-\tau/2} b(e^\tau)_{\tau \geq 0}.$$

This construction of an Ornstein-Uhlenbeck process consists of the operation

$$(b(t), t \geq 0) \rightarrow (t^{-1/2} b(t), t \geq 1)$$

followed by the operation

$$(t^{-1/2} b(t), t \geq 1) \rightarrow ((e^\tau)^{-1/2} b(e^\tau), \tau \geq 0).$$

The first of these operations re-normalises b so that it has an initial (and stationary) distribution, given by a standard Gaussian distribution. The process thus obtained then fails to be stationary; hence, in the second operation, we re-normalise the time-parameter t to $\tau = \log t$, so that the new process is stationary, i.e. the transition density $p_{\sigma\tau}(x, y) = P[X(\tau) \in dy | X(\sigma) = x]$ of the process $X(\tau)$ depends on (σ, τ) only via $\tau - \sigma$. Indeed, from the definition of X it is easily verified that the common value of $p_{\sigma(\sigma+\tau)}(x, y)$ for all σ is given by

$$p_\tau(x, y) = (1 - e^{-\tau})^{-1/2} \varphi((y - xe^{-\tau/2}) / (1 - e^{-\tau})^{1/2})$$

where φ is the standard normal density function

$\varphi(x) = (2\pi)^{-1/2} e^{-x^2/2}$. The transition density $p_\tau(x, y)$ is the same

as the density function of the value at τ of the process

$$X_x(\tau) = e^{-\tau/2}(x + b(e^\tau - 1)).$$

This is a convenient representation of an Ornstein-Uhlenbeck process starting at x in terms of a Brownian motion.

1.2 Hermite polynomials. The *Hermite polynomials* $H_n(x)$ are given by the generating function

$$e^{tx - t^2/2} = \sum_{n=1}^{\infty} H_n(x) t^n$$

where the exponential on the left is a formal power series in t . It can then be verified that the functions $((n!)^{1/2} H_n(\cdot), n \geq 0)$ form an ortho-normal basis for $L^2(\mathbb{R}, \varphi(x)dx)$.

1.3 The Ornstein-Uhlenbeck semigroup. We now derive the semigroup $(T_t)_{t \geq 0}$ associated with the Ornstein-Uhlenbeck transition function. This is a semigroup of contractions T_t on $L^2(\mathbb{R}, \varphi(x)dx)$ given by

$$(T_t f)(x) = \int_{-\infty}^{\infty} f(y) p_t(x, y) dy = E[f(X_x(t))].$$

It is easily verified that the T_t are contractions, using the fact that $\varphi(x)dx$ is the stationary measure of the Ornstein-Uhlenbeck process.

Consider the function

$$f_\lambda(x) = e^{i\lambda x + \lambda^2/2}.$$

Then by the generating function of H_n ,

$$f_\lambda(\cdot) = \sum_0^\infty (i\lambda)^n H_n(\cdot) \quad (1)$$

where, since the $L^2(\varphi(x)dx)$ - norm of $H_n(\cdot)$ is $(n!)^{-1/2}$, the above convergence (1) holds in an $L^2(\varphi(x)dx)$ sense (and also almost everywhere (dx)). Hence, using the above characterisation of $X_x(t)$ in terms of Brownian motion, we have

$$T_t f_\lambda(x) = e^{\lambda^2/2} E(e^{i\lambda Z})$$

where Z is a $N(e^{-t/2}, 1-e^{-t})$ random variable. Hence

$$T_t f_\lambda(x) = \exp\{i\lambda e^{-t/2}x + (\lambda^2 e^{-t})/2\} = f\left[\lambda e^{-t/2}\right](x).$$

Again using the generating function of H_n , we have

$$T_t f_\lambda(\cdot) = \sum_0^\infty (i\lambda e^{-t/2})^n H_n(\cdot), \quad (2)$$

while by applying T_t to (1) we have

$$T_t f_\lambda(\cdot) = \sum_0^\infty (i\lambda)^n T_t H_n(\cdot). \quad (3)$$

(2) and (3) each hold in an $L^2(\varphi(x)dx)$ sense (indeed the sum of the L^2 norms converge in each case) for all real λ . Hence, comparing coefficients of λ^n we have

$$T_t H_n(x) = e^{-nt/2} H_n(x) \quad \text{almost all } x \in \mathbb{R}. \quad (*)$$

Since T_t is a bounded operator on $L^2(\varphi(x)dx)$ and $\{H_n, n \geq 0\}$ form a basis for this space, (*) determines the operator T_t completely.

1.4 The Brownian sheet. A *two-parameter Wiener Process* taking values in \mathbb{R}^d is a random function

$$w: \mathbb{R}_+^2 \rightarrow \mathbb{R}^d$$

$$w(\tau, t) = (w_1(\tau, t), w_2(\tau, t), \dots, w_d(\tau, t))$$

defined on some underlying probability space (Ω, Σ, P) , such that the sample paths of w are continuous, the finite-dimensional distributions of w are all zero-mean Gaussian distributions, and the covariance structure of w is given by

$$E [w_i(\tau, t) w_j(\sigma, s)] = \min(\tau, \sigma) \min(t, s) \delta_{ij}.$$

An N - parameter *wiener process* taking values in \mathbb{R}^d is defined similarly. We shall refer to a two-parameter Wiener process in \mathbb{R}^d as a *Brownian sheet* in \mathbb{R}^d . Later, we shall outline a construction of a Brownian sheet. Denote by $Q(\tau, t)$ the rectangle $[0, \tau] \times [0, t]$; $w(\tau, t)$ may be thought of as the integral of white noise over this rectangle.

1.5 Properties of the Brownian sheet. Let $w(\tau, t)$ be a two-parameter Wiener process taking values in \mathbb{R}^d .

Scaling property. If we fix $\tau_0 \in (0, \infty)$ then the process

$$\tau_0^{-1/2} w(\tau_0, \cdot)$$

is a Brownian motion in \mathbb{R}^d .

Strong Markov property. Let Σ_τ denote the σ -algebra generated by $\{w(\sigma, t) : (\sigma, t) \in (0, \tau) \times (0, \infty)\}$. Let $(\Sigma_\tau^+, \tau \geq 0)$ denote the right continuous modification of the filtration $(\Sigma_\tau, \tau \geq 0)$. If τ_0 is a Σ_τ^+ -stopping time, the process

$$w(\tau_0 + \tau, t) - w(\tau_0, t), \quad \tau \geq 0, t \geq 0$$

is another two-parameter Wiener process, and is independent of Σ_τ^+ .

This fact can be proved in a similar way to Theorem 1 of Walsh (1982).

Hölder Continuity. Lévy's Hölder continuity of Brownian Motion extends to the two-parameter Wiener process as follows:

1.6 THEOREM. (Orey, Pruitt (1973)). For $x > 0$, Set

$h(x) = (2x|\log x|)^{1/2}$. For τ and t in \mathbb{R}_+ , Set

$Q(\tau, t) = [0, \tau] \times [0, t]$. For $\underline{s} = (\sigma, s)$ and $\underline{t} = (\tau, t)$ in \mathbb{R}_+^2 ,

denote by $D(\underline{s}, \underline{t})$ the area of the symmetric difference between $Q(\underline{s})$ and $Q(\underline{t})$. Then

$$\lim_{\varepsilon \downarrow 0} \sup \{ |w(\underline{t}) - w(\underline{s})| / h(D(\underline{s}, \underline{t})) : \underline{s}, \underline{t} \in Q(1, 1), D(\underline{s}, \underline{t}) < \varepsilon \} = 2^{1/2}$$

almost surely.

1.7 The Ornstein Uhlenbeck process in Wiener space. Let W_0^d denote

d -dimensional Wiener space, the space of all continuous paths

$B: [0, \infty) \rightarrow \mathbb{R}^d$ vanishing at zero, with the topology of uniform

convergence on compact intervals. Let W be Wiener measure on the

Borel sets in W_0^d . Let $(w(\tau, t))_{\tau \geq 0, t \geq 0}$ be the Brownian sheet in \mathbb{R}^d . As τ increases, the section $w(\tau, \cdot)$ can be thought of as an evolving element of W_0^d , a "Wiener process in Wiener space". An *Ornstein-Uhlenbeck process* in W_0^d , denoted $(B_\tau)_{\tau \geq 0}$, with initial distribution given by W , can be constructed (see Meyer (1980)) analogously to the Ornstein-Uhlenbeck process on the line, by setting

$$B_\tau(\cdot) = e^{-\tau/2} w(e^\tau, \cdot) \quad \tau \geq 0. \quad (**)$$

By the scaling property of the Brownian sheet (section 1.5), W is the initial and stationary distribution of $(B_\tau)_{\tau \geq 0}$.

Thus the path $B_\tau = B_\tau(\cdot)$ is an element of W_0^d , and B_τ travels in W_0^d as τ varies. We say that a property of Brownian motion in \mathbb{R}^d holds *quasi-everywhere* with respect to the Ornstein-Uhlenbeck process in \mathbb{R}^d if

$$P[B_\tau \in E, \tau \geq 0] = 1$$

where E is the set of paths in W_0^d with that property. We then say that $W_0^d \setminus E$ is *exceptional* with respect to the Ornstein-Uhlenbeck process on Wiener space.

Instead of B_τ we shall often find it more convenient to consider the path $b_\tau = b_\tau(\cdot)$, given by

$$b_\tau(\cdot) = w(\tau, \cdot), \quad \tau \geq 1.$$

(**) enables us easily to translate results on B_τ to corresponding results on b_τ .

The above (probabilistic) definition of an exceptional set in Wiener space is the one we shall use in our proofs of quasi-everywhere results. For the sake of a more complete view of the meaning of the term "quasi-everywhere", we shall in the rest of this chapter derive the infinitesimal generator of the Ornstein-Uhlenbeck process on Wiener space, and hence obtain a characterisation of the exceptional sets in terms of a Dirichlet form on Wiener space. This is by now fairly standard (see for example Strook (1981), Meyer (1982), Fukushima (1984), Watanabe (1984)).

We shall work only on $C([0,1] \rightarrow \mathbb{R})$ here; the generalisation to $C([0,\infty) \rightarrow \mathbb{R}^d)$ is straightforward.

1.8 Construction of the Brownian Sheet. Consider the

Lévy-Ciesielski construction of Brownian motion using the Haar series of functions f_n (see McKean (1969)). In this construction the sample path $b(\cdot)$ is given by

$$b(t) = \sum_{n=0}^{\infty} Y_n f_n(t)$$

where the coefficients Y_n of the Haar functions f_n are i.i.d. standard normals. Conversely the sample path $b(\cdot)$ determines the random variables Y_n almost surely by the Itô integral

$$Y_n = \int_0^{\infty} F_n(t) b(dt).$$

Thus we have a measure space isomorphism between

$(C[0,1], W)$ and $(\mathbb{R}^{Z_+}, \Gamma)$, where Γ is an infinite product of the standard Gaussian measure $\varphi(x)dx$. This determines a Hilbert space

isomorphism between $L^2(W)$ and $L^2(\Gamma)$; the latter is known as *Fock space*.

We now allow the coefficients Y_n of the Haar functions f_n to vary in a second "time" parameter τ , according to mutually independent Brownian motions (respectively Ornstein-Uhlenbeck processes). The resulting process $w(\tau, t) = \sum_{n=0}^{\infty} Y_n(\tau) f_n(t)$ is a two-parameter Wiener process (respectively an Ornstein-Uhlenbeck process in Wiener space). To check this we need only check the covariance functions (straightforward) and the joint continuity in (τ, t) (much as in the case of the construction of Brownian motion; see Csorgo and Revesz (1981), pages 59-61 for details).

1.9 The Ornstein-Uhlenbeck Semigroup on Fock Space. By our construction of the Brownian sheet, the Ornstein-Uhlenbeck process in Wiener space corresponds to an Ornstein-Uhlenbeck process in $\mathbb{R}^{\mathbb{Z}_+}$, i.e. a sequence of independent Ornstein-Uhlenbeck processes on the line.

Denote by $(P_t, t \geq 0)$ the semigroup of operators on $L^2(\Gamma)$ (Fock space) associated with the Ornstein-Uhlenbeck process on $\mathbb{R}^{\mathbb{Z}_+}$.

Denote by \mathcal{N} the set of all multi-indices $\underline{n} = (n_1, n_2, n_3, \dots)$, where the n_i are non-negative integers, and n_i is non-zero for only finitely many i . \mathcal{N} is countable. For $\underline{n} \in \mathcal{N}$ set

$$|\underline{n}| = \sum_{i=1}^{\infty} n_i < \infty.$$

For $\underline{n} \in \mathcal{N}$ define the "Hermite multinomial" $H_{\underline{n}} \in L^2(\mathbb{R}^{Z_+}, \Gamma)$ by

$$H_{\underline{n}}(x_1, x_2, \dots) = \prod_{i=1}^{\infty} [(n_i!)]^{1/2} H_{n_i}(x_i)$$

($H_0 \equiv 1$ so only finitely many terms in this product are not 1). The class $\{H_{\underline{n}} : \underline{n} \in \mathcal{N}\}$ forms a (countable) ortho-normal basis for $L^2(\Gamma)$.

For $\underline{n} \in \mathcal{N}$ we have for $x = (x_1, x_2, x_3, \dots) \in \mathbb{R}^{Z_+}$

$$P_t H_{\underline{n}}(\underline{x}) = E_{\underline{x}} \prod_{i=1}^{\infty} [(n_i!)]^{1/2} H_{n_i}(X^{(i)})$$

where the $X^{(i)}$ ($i \geq 1$) are mutually independent Ornstein-Uhlenbeck processes on the line such that $X^{(i)}$ starts at x_i for each i .

Hence, by the expression (*) (from section 1.3) for the operator T_t associated with an Ornstein-Uhlenbeck process in \mathbb{R} ,

$$\begin{aligned} P_t H_{\underline{n}}(\underline{x}) &= \prod_{i=1}^{\infty} [(n_i!)]^{1/2} T_t H_{n_i}(x_i) \\ &= e^{-|\underline{n}|t/2} H_{\underline{n}}(\underline{x}). \end{aligned} \quad (***)$$

Since the $H_{\underline{n}}$, $\underline{n} \in \mathcal{N}$ form an ortho-normal basis for $L^2(\Gamma)$, (***)

determines P_t completely. Moreover, P_t is clearly symmetric

since $\{H_{\underline{n}}, \underline{n} \in \mathcal{N}\}$ forms an ortho-normal basis for $L^2(\Gamma)$

consisting of eigenvectors for P_t .

1.10 The generator of the Ornstein-Uhlenbeck process. We now describe the generator $(A, \mathcal{D}(A))$ of the semigroup P_t . This is defined (see Fukushima (1980), section 1.3) to be the derivative at $t=0$ of the operator-valued function P_t in the sense that the operator A with domain $\mathcal{D}(A) \subset L^2(\Gamma)$ is given by

$$A(f) = L^2\text{-}\lim_{t \rightarrow 0} [t^{-1}(P_t f - f)]$$

$$\mathcal{D}(A) = \{f \in L^2(\Gamma) : L^2\text{-}\lim_{t \rightarrow 0} [t^{-1}(P_t f - f)] \text{ exists}\}.$$

Since the derivative at $t = 0$ of $e^{-\alpha t}$ is $(-\alpha)$, the expression (*) from section (1.3) suggests the following:

1.11 PROPOSITION. the generator of P_t is given by

$$A\left(\sum_{\underline{n} \in \mathcal{N}} a_{\underline{n}} H_{\underline{n}}\right) = -\frac{1}{2} \sum_{\underline{n} \in \mathcal{N}} |\underline{n}| a_{\underline{n}} H_{\underline{n}}$$

$$\mathcal{D}(A) = \{f = \sum_{\underline{n} \in \mathcal{N}} a_{\underline{n}} H_{\underline{n}} \in L^2 : \sum_{\underline{n} \in \mathcal{N}} |\underline{n}|^2 a_{\underline{n}}^2 < \infty\}.$$

Proof. Suppose $f = \sum_{\underline{n} \in \mathcal{N}} a_{\underline{n}} H_{\underline{n}} \in \mathcal{D}(A)$. Then

$$\left\| \frac{P_t f - f}{t} - Af \right\|_2^2 = \sum_{\underline{n} \in \mathcal{N}} a_{\underline{n}}^2 \left(\frac{e^{-|\underline{n}|t/2} - 1}{t} + |\underline{n}|/2 \right)^2 \quad (1)$$

each term in this summation tends to zero, and since for $\alpha > 0$ $1 - e^{-\alpha t} \leq \alpha t$ for all $t > 0$, the $\underline{n}^{\text{th}}$ term is dominated by

$$a_{\underline{n}}^2 |\underline{n}|^2.$$

Hence by Lebesgue's dominated convergence theorem on the countable discrete measure space \mathcal{N} , the right hand side of (1) converges to zero as $t \rightarrow 0$, as required.

Conversely, if $f \notin \mathcal{D}(A)$ it is easily seen that

$$\left\| \frac{P_t f - f}{t} \right\|_2 \rightarrow \infty \text{ as } t \rightarrow 0, \text{ so that there is no } L^2\text{-limit for } t^{-1}(P_t f - f).$$

□

1.12 Wiener Chaos. By the equivalence of $L^2(W)$ to Fock space,

$\{H_{\underline{n}}(\int f_i dw, i \geq 1), n \in \mathcal{N}\}$ forms an ortho-normal basis for $L^2(W)$. Define

Z_n to be the linear span of $\{H_{\underline{n}}(\int f_i dw, i \geq 1) : |\underline{n}| = n\}$. Z_n is known as

"Wiener chaos of order n ". Thus $L^2(W)$ decomposes into a direct

sum of orthogonal spaces Z_n . This is known as the Wiener-Itô

decomposition. It should be noted that the space Z_n does not

depend on our use of the Haar functions. For it can be proved (see

Strook (1981), page 409) that Z_n is the space of all elements F

of $L^2(W)$ which may be expressed by the n -ple Itô integral

$$F(w) = \int_0^1 \int_0^{t_1} \int_0^{t_2} \dots \int_0^{t_{n-1}} f(t_1, t_2, \dots, t_n) b(dt_n) b(dt_{n-1}) \dots b(dt_1).$$

By Proposition 1.11 and the equivalence of $L^2(W)$ with $L^2(\Gamma)$, we

have the following well-known result:

1.13 THEOREM. Let P_n be the projection of $L^2(W)$ onto Wiener chaos of order n . The infinitesimal generator of the Ornstein-Uhlenbeck process on Wiener space is given by

$$\mathcal{D}(A) = \{ F \in L^2(W) : \sum_{n=1}^{\infty} n^2 \| P_n F \|^2_{L^2(W)} < \infty \}.$$

$$A(F) = \sum_{n=1}^{\infty} n P_n(F)$$

1.14 An associated Dirichlet form and capacity on Wiener space. We have derived the generator $(A, \mathcal{D}(A))$ of the Ornstein-Uhlenbeck process on Wiener space. As in section 1.3 of Fukushima (1980), this gives us an associated Dirichlet form $(\mathcal{E}_1, \mathcal{D}(\mathcal{E}_1))$ on $L^2(W)$, given by

$$\mathcal{D}(\mathcal{E}_1) = \{ F \in L^2(W) : \sum_{n=1}^{\infty} n \| P_n(f) \|^2_2 < \infty \}$$

$$\mathcal{E}_1(F, G) = \sum_{n=1}^{\infty} n (P_n F, P_n G) + (F, G)$$

where (\cdot, \cdot) denotes the $L^2(W)$ inner product. We also have an associated capacity on subsets of $C[0,1]$, given for open E by

$$\text{Cap}_1(E) = \inf \{ \mathcal{E}_1(F, F) : F \in \mathcal{D}(\mathcal{E}_1), F \geq 0 \text{ on } E \}$$

and for general E by

$$\text{Cap}_1(E) = \inf \{ \text{Cap}_1(E^o), E^o \text{ open}, E^o \supset E \}.$$

We wish to use Theorem 4.3.1 of Fukushima (1984) to equate sets of zero capacity with those which are never visited by the

Ornstein-Uhlenbeck process on Wiener space. The theory is not

directly applicable because W_0^d is not a locally compact space.

However, W_0^d is a Polish space, and the proofs in Fukushima (1980)

hold equally well (In particular, a Borel subset of a Polish space is capacitable; see Boubaki (1974), Chapter IX, section 6, Définition 2, Proposition 10 and Théorème 6). Hence we have the following result (see Fukushima (1984), page 164):

1.15 THEOREM. Let E be a Borel subset of W_0^d . Then E is exceptional with respect to the Ornstein-Uhlenbeck process in Wiener space if and only if $\text{Cap}_1(E) = 0$.

An alternative method to show that the two notions of a set of zero capacity coincide is to embed Wiener space in a locally compact space so as to be able to apply the theory in Fukushima (1980) directly. For details see Takeda (1984), section 6. Further background material on the theory of Dirichlet forms on spaces which are not locally compact can be found in Kusuoka (1982) and in Bouleau and Hirsch (1988).

In this chapter we recall the definition of the local time of a (not necessarily random) function from an arbitrary measure space (here, usually a subset of \mathbb{R}^N) to \mathbb{R}^d , and show how we may use the local time to estimate Hausdorff measures of the level sets of a continuous function. We then recall a useful Fourier-analytic expression of the local time of a Gaussian random function. Further background information (and a fuller bibliography) on local times can be found in Geman and Horowitz (1980).

The following definitions refer to a general measure space; in practice, this will always be a Borel subset of Euclidean space with Lebesgue measure.

2.1 Local time: definitions. Let (E, Σ, m) be a measure space.

Suppose that $X: E \rightarrow \mathbb{R}^d$ is a Borel measurable function, and $B \in \Sigma$ is a measurable set in E . The *occupation measure* of X relative to B is the measure μ_B on the Borel sets in \mathbb{R}^d induced by the restriction of X to B , i.e.

$$\mu_B(A) = m(B \cap X^{-1}(A)),$$

We say that X has a *local time relative to* B (or more concisely, X has a *local time in* B) if μ_B is absolutely continuous with respect to Lebesgue measure λ^d . In this case, we define the *local time relative to* B (or the *local time in* B) of X by

$$\alpha(x, B) = (d\mu_B/d\lambda^d)(x) \quad x \in \mathbb{R}^d,$$

so $\alpha(\cdot, B)$ is defined only a.e. $d\lambda^d(x)$. A version of the local

time is a particular choice of α . When X has a local time α , we have by definition

$$\int_{t \in B} f(X(t)) dt = \int_{\mathbb{R}^d} f(x) \cdot \alpha(x, B) dx, \quad f \in C_0(\mathbb{R}^d) \quad (*)$$

and this property determines the local time α (a.e. $d\lambda^d$).

2.2 Intersection local time: definitions. Suppose that for

$i = 1, 2$, (E_i, Σ_i, m_i) are measure spaces and $X_i: E_i \rightarrow \mathbb{R}^d$ are Borel measurable functions. Suppose that B is a product measurable subset of $E_1 \times E_2$. and the function $X: E_1 \times E_2 \rightarrow \mathbb{R}^d$ given by

$$X(x_1, x_2) = X_2(x_2) - X_1(x_1)$$

has a local time relative to B . Then we shall refer to this local time as the *intersection local time* of X_1 and X_2 relative to B .

Suppose that in the above, the measure spaces (E_1, Σ_1, m_1) and (E_2, Σ_2, m_2) are in fact the same space, and that $X_2 \equiv X_1$. Then we shall refer to the intersection local time of X_1 and X_1 relative to B as the *self-intersection local time* of X_1 relative to B .

2.3 Hausdorff measures: definitions. Suppose that $h(s)$ is an increasing, right continuous real-valued function defined on $\{s \geq 0\}$, with $h(0) = 0$. Then we shall call h a *Hausdorff measure function*. Given such h , and a set E in \mathbb{R}^N (or any metric space), then the *Hausdorff h -measure* of E is given by

$$h\text{-meas}(E) = \lim_{\epsilon \downarrow 0} \inf \left\{ \sum_{i \geq 1} h(\text{diam } A_i) \right\},$$

the infimum being taken over all countable collections $(A_i, i \geq 1)$ of sets in \mathbb{R}^N of diameter less than ϵ which cover E . For real

$\alpha \geq 0$, set $h^\alpha(x) = x^\alpha$ ($x \geq 0$). The Hausdorff dimension of E is given by

$$\dim E = \inf\{\alpha \geq 0: h^{\alpha-m}(E)=0\}.$$

2.4 Application of local time to Hausdorff measures of level sets.

Lévy introduced the local time of Brownian motion on the line as a measure on its level sets. We use the concept of local time for similar purposes, as an estimate for Hausdorff measure functions of level sets. In the next two lemmas, let B_0 be a bounded open set in \mathbb{R}^N and let $X: B_0 \rightarrow \mathbb{R}^d$ be a continuous (non-random) function with a local time relative to B_0 . Then for each Borel subset B of B_0 , X has a local time relative to B , which we denote by $\alpha(\cdot, B)$.

For $x \in \mathbb{R}^d$, define the level set M_x of X at x by $M_x = \{t \in B_0: X(t) = x\}$. It is well known that if $\alpha(x, B)$ satisfies a Hölder condition in the set variable B then we can obtain a lower bound for an appropriate Hausdorff measure of M_x as follows:

2.5 LEMMA. Let h be a Hausdorff measure function. Suppose that versions of the local times $(\alpha(x, B), x \in \mathbb{R}^d)$, defined for each Borel $B \subset B_0$, can be chosen so that

- (i) $\alpha(x, B)$ is continuous in x for each rectangular B , and
- (ii) For each $x \in \mathbb{R}^d$, $\alpha(x, \cdot)$ is a finite measure on the Borel subsets of B_0 , and satisfies

$$\alpha(x, B) \leq \text{const.} \cdot h(\text{diam } B) \quad \text{all Borel } B \subset B_0.$$

Then

$$h-m(M_x) \geq \text{const.} \cdot \alpha(x, B_0).$$

Proof. For each $x \in \mathbb{R}^d$ the measure $\alpha(x, dt)$ is supported by the level set M_x . Indeed, given any compact rectangle R in the open set $B_0 \setminus M_x$, the occupation measure μ_R is supported by the compact set $X(R)$ which is disjoint from some neighbourhood of x . Since $\alpha(\cdot, R)$ is the continuous version of the Radon-Nikodym derivative of μ_R , $\alpha(x, R)$ must be zero; hence $\alpha(x, dt)$ is supported by M_x , as we asserted. The rest of the proof follows directly from the definition of Hausdorff measure. \square

Conversely, if X satisfies a Hölder condition in t , we may find an upper bound for an appropriate Hausdorff measure of M_x :

2.6 LEMMA. (Essentially Adler (1978), Lemma 7.) Let g be a Hausdorff measure function such that the function h given by

$$h(t) = t^N (g(t^{N/2}))^{-d}$$

is also a Hausdorff measure function. Suppose that for some compact $K \subset B_0$ and some $\delta > 0$, the function X satisfies

$$|X(t) - X(s)| \leq g(|t - s|) \quad (s, t \in B_0, |t - s| < \delta)$$

(1)

and that $\alpha(x, K)$ can be chosen to be continuous in x .

Then,

$$h\text{-}m(M_x \cap K) < \infty, \quad x \in \mathbb{R}^d.$$

An immediate consequence is the following:

2.7 COROLLARY. If $0 \leq \gamma \leq N/d$ and X is Hölder continuous of any order less than γ , then

$$\dim(M_x) \leq N - \gamma d$$

2.8 Proof of Lemma 2.6. By the Hölder condition (1) and the compactness of $M_x \cap K$, for small ϵ we may cover $M_x \cap K$ by those closed cubes $C \subset B_0$ of the form $\prod_{i=1}^N [m_i \epsilon, (m_i + 1) \epsilon]$, with m_i integers, such that $|X(t) - x| \leq g(\epsilon 2^{N/2})$, all $t \in C$. Let $\#(\epsilon)$ be the number of such cubes. Then $\epsilon^N \#(\epsilon)$ is at most the occupation measure of the ball at x of radius $g(\epsilon 2^{N/2})$; hence, by the continuity of the local time α ,

$$\epsilon^N \#(\epsilon) \leq (g(\epsilon 2^{N/2}))^d (c + o(1)) \quad (\epsilon \rightarrow 0)$$

where c equals the local time at x times the volume of the unit ball. Hence, by definition of Hausdorff h -measure,

$$\begin{aligned} h\text{-}m(M_x) &\leq \liminf_{\epsilon \rightarrow 0} \left[\#(\epsilon) h(\epsilon) \right] \\ &= \liminf_{\epsilon \rightarrow 0} \left[\#(\epsilon) (g(\epsilon 2^{N/2}))^{-d} \epsilon^N \right] \\ &< \infty. \end{aligned}$$

□

2.9 Fourier analysis of local times of stochastic processes. So far

we have considered only local times of a deterministic function X from \mathbb{R}^N to \mathbb{R}^d . We now consider the case where X is random; i.e. X is an N -parameter stochastic process taking values in \mathbb{R}^d . There are by now several ways to consider the continuous version of the local times of X (if they exist) when X is Gaussian (see Geman and Horowitz (1980)). Generally, these use "local non-determinism", a property which fails to hold for the processes we shall be most interested in. To avoid this problem, we find it clearest to use an older result due to Berman (1969) (generalised here to $d \neq 1$).

The result says that under certain conditions, we can almost surely obtain a version of the local time of X by formally

inverting the Fourier transform of the occupation measure, giving us an expression of the local time as a singular integral whose continuity we can check by Kolmogorov's lemma. Typically, this gives us a nasty integral to estimate, but it turns out that we can estimate this in the cases we consider here. The following two lemmas express Berman's results.

2.10 LEMMA. (i) Let B be a Borel set in \mathbb{R}^N and let X be an N - parameter Gaussian stochastic process taking values in \mathbb{R}^d . Let

$$\mu_B^*(u) = \int_{\underline{t} \in B} \exp(iu \cdot X(\underline{t})) d\underline{t} \quad (u \in \mathbb{R}^d)$$

be the Fourier transform of the occupation measure of X relative to B (cf. equation (*) of section 2.1). Suppose that $\mu_B^*(\cdot)$ is almost surely square integrable. Then X has a local time in B almost surely.

(ii) Suppose in addition to the above that for some $k \in \mathbb{Z}_+$

$$\int_{(\mathbb{R}^d)^{2k}} \int_{B^{2k}} E \left[\exp \left\{ i \sum_{j=1}^{2k} u_j \cdot X(\underline{t}_j) \right\} \right] d\underline{t}_1 \dots d\underline{t}_{2k} du_1 \dots du_{2k} < \infty \quad (**)$$

(Note that our process is Gaussian so the integrand is real and positive). Then there exists a separable real-valued process

$\Psi(x) = \Psi(x, \omega) \quad (x \in \mathbb{R}^d, \omega \in \Omega)$, such that

$$E |\Psi(x) - \Psi_m(x)|^{2k} \rightarrow 0 \quad (n \rightarrow \infty), \quad \text{uniformly in } x,$$

where $\Psi_m(x)$ is given by

$$\begin{aligned} \Psi_m(x) &= (2\pi)^{-d} \int_{|u| \leq m} \mu_B^*(u) e^{-iu \cdot x} du \\ &= (2\pi)^{-d} \int_{|u| \leq m} \int_{\underline{t} \in B} \exp(iu \cdot (X(\underline{t}) - x)) d\underline{t} du. \end{aligned}$$

Proof. Part (i) is routine harmonic analysis; a measure is determined by its Fourier transform, so if this is in $L^2(dx)$, the inverse Fourier transform on L^2 will give us the derivative of the measure. As for part (ii), observe that $\Psi_m(x)$ is real for all x and n , so that

$$E|\Psi_m(x) - \Psi_n(x)|^{2k} = E(\Psi_m(x) - \Psi_n(x))^{2k}.$$

By (**) and the definition of Ψ_m , we find that the above expression tends to zero as $n, m \rightarrow \infty$, uniformly in x . Hence, for each x we can find a random variable $\Psi(x)$, such that

$$E|\Psi_M(x) - \Psi(x)|^{2k} \rightarrow 0 \quad (m \rightarrow \infty) \quad \text{uniformly in } x.$$

(1)

Take a separable modification of this (Doob (1953), Theorem 2.4 (page 57) extended to $d \neq 1$), also denoted Ψ . Then (1) still holds. \square

2.11 Remarks. We may express Ψ formally by the improper integral

$$\Psi(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \mu_B^*(u) \cdot e^{-iu \cdot x} du.$$

If $\Psi(x, \omega)$ is continuous in x for almost all ω , then it is jointly measurable in (x, ω) (Doob (1953), Theorem 2.5 (page 60) extended to $d \neq 1$). Hence the following result is applicable:

2.12 LEMMA. If Ψ is jointly measurable in (x, ω) , then for almost all ω , $\Psi(\cdot, \omega)$ is a version of the local time relative to B of X .

Proof. Let A be a bounded Borel set in \mathbb{R}^d . Then by Fubini's theorem (and the joint measurability of Ψ) and Hölder's inequality,

$$\begin{aligned} E \int_A |\Psi_m(x) - \Psi(x)| dx &\leq \int_A (E |\Psi_m(x) - \Psi(x)|^{2k})^{1/2k} dx \\ &\rightarrow 0 \quad (m \rightarrow \infty). \end{aligned}$$

Also the almost sure square-integrability of $\mu^*(\cdot)$ ensures (by classical Fourier analysis) that the functions $\Psi_m(\cdot)$ converge in $L^2(dx)$ to $d\mu/d\lambda^d$, almost surely, so that by Hölder's inequality,

$$\int_A \Psi_m(x) dx \rightarrow \mu_B(A) \quad (m \rightarrow \infty).$$

So the random variables $\int_A \Psi_m(x) dx$ converge in $L^1(dP)$ to $\int_A \Psi(x) dx$ and converge almost surely to $\mu_B(X)$. Hence the two limiting random variables are the same almost surely, i.e.

$$\int_A \Psi(x) dx = \mu_B(A)$$

(1)

almost surely. Hence for almost all ω , (1) holds for all bounded rectangles A with corners with rational co-ordinates; by a monotone class argument on the class of sets A for which (1) holds, we see that (1) holds for all Borel A in \mathbb{R}^d , and this gives us the result. \square

2.13 Continuity of the local time and the integral $J_X(k, \gamma, B)$.

The above expression Ψ of the local time of an N -parameter Gaussian process X in \mathbb{R}^d is useful for checking the continuity of the local time by applying Kolmogorov's lemma to the expression of Ψ as an improper integral. To do this, let $k \in \mathbb{Z}_+$ and assume for the moment that Lemma 2.10 is applicable, so our candidate for the

local time of X in B is given by

$$\Psi(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \left[\int_B \exp\{iu \cdot (X(t) - x)\} dt \right] du$$

in the sense of Lemma 2.10. As Ψ is real, for x and y in \mathbb{R}^d

$$\begin{aligned} E|\Psi(x) - \Psi(y)|^{2k} &= E[\Psi(x) - \Psi(y)]^{2k} \\ &= E \left[\int_{u \in \mathbb{R}^d} \int_{t \in B} (e^{-iu \cdot x} - e^{-iu \cdot y}) \exp\{iu \cdot X(t)\} dt du \right]^{2k}. \end{aligned}$$

Here the improper integral over \mathbb{R}^d is the L^{2k} limit of a sequence of real-valued random variables given by integrals over finite regions (see Lemma 2.10). Hence we may take the expectation inside the multiple integral, in the sense that $E|\Psi(x) - \Psi(y)|^{2k}$ is a "principal value" of the improper integral.

$$\int_{(\mathbb{R}^d)^{2k}} \left[\int_B \left[\prod_{\ell=1}^{2k} \{\exp(-iu_\ell \cdot x) - \exp(-iu_\ell \cdot y)\} \right] E \exp\{i \sum_{j=1}^{2k} u_j \cdot X(t_j)\} d\bar{t} \right] d\bar{u}.$$

where $\bar{u} = (u_1, \dots, u_{2k})$, $\bar{t} = t_1, \dots, t_{2k}$. But For $0 \leq \gamma < 1$, we have

$$|\exp(-iu_\ell \cdot x) - \exp(-iu_\ell \cdot y)| \leq 2|u_\ell|^\gamma |y - x|^\gamma,$$

so

$$E|\Psi(x) - \Psi(y)|^{2k} \leq 2^{2k} |y - x|^{2k\gamma} J_X(2k, \gamma, B) \quad (1)$$

where (writing $\text{Var}(Z)$ for the variance of a random variable Z) we define the integral $J_X(2k, \gamma, B)$ by

$$\begin{aligned} J_X(k, \gamma, B) &= \int_{\bar{u} \in \mathbb{R}^{dk}} \int_{\bar{t} \in B^k} \left[\prod_{\ell=1}^k |u_\ell|^\gamma \right] \exp\{-(\frac{k}{2}) \text{Var}(\sum_{j=1}^k u_j \cdot X(t_j))\} d\bar{t} d\bar{u} \\ &= \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} E \prod_{j=1}^k \{|u_j|^\gamma \mu_B^*(u_j)\} du_1 \dots du_k. \end{aligned} \quad (***)$$

(so the integrand in the definition of J_X is real and positive).

Thus if for some k and γ with $2k\gamma > d$, $J_X(2k, \gamma, B)$ is finite, then our estimate for $|e^{-iu \cdot x} - e^{-iu \cdot y}|$ has given us the

Kolmogorov-type condition

$$E|\Psi(x)-\Psi(y)|^{2k} \leq c |x-y|^{2k\gamma}$$

allowing us to deduce continuity of $\Psi(x)$ in x (and by Remark 2.11 and Lemma 2.12, this means Ψ is a version of the local time).

Moreover, the integral which is required to be finite in the hypothesis for part (ii) of Lemma 2.10 is just $J_X(2k,0,B)$. Thus, we need only check the square integrability of the Fourier transform of occupation measure (see part (i) of lemma 2.10), together with finiteness conditions on J_X , to be able to apply Lemmas 2.10 and 2.12. We may summarise the above discussion (similar to that in section 2 of Geman, Horowitz and Rosen (1984)) as follows:

2.14 THEOREM. Suppose that X is a Gaussian process in \mathbb{R}^d defined on a Borel set B in \mathbb{R}^N . Suppose that the Fourier transform μ_B^* of occupation measure is square integrable over \mathbb{R}^d . Suppose also that for some $\gamma > 0$ and strictly positive integers k and k' with $k\gamma > d$, we have

$$J_X(2k,\gamma,B) < \infty$$

and

$$J_X(2k',0,B) < \infty.$$

Then X has a local time in B with a continuous version Ψ which may be expressed formally by the improper integral

$$\Psi(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \left[\int_B \exp\{iu \cdot (X(t)-x)\} dt \right] du.$$

2.15 Remark. The above is of no use without being able to estimate J_X . As we shall see below, most of the work on estimating J_X in our main example has already been done for us by Geman, Horowitz and Rosen (1984).

Before giving examples where we may apply Theorem 2.13, we note that $J_X(2k, \gamma, \cdot)$ is countably subadditive in an L^{2k} sense:

2.16 LEMMA. Suppose that X is a Gaussian, N -parameter, \mathbb{R}^d -valued process. For every $k \in \mathbb{Z}_+$ and $\gamma \geq 0$, $(J_X(2k, \gamma, \cdot))^{1/2k}$ is a countably subadditive set function.

Proof. By the definition (***) of J_X (section 2.13),

$$J_X(2k, \gamma, B) = \lim_{n \rightarrow \infty} E \left[\int_{|u| \leq n} |u|^\gamma \mu_B^*(u) du \right]^{2k}.$$

The integral of $|u|^\gamma \mu_B^*(u)$ over $\{|u| \leq n\}$ is real for all n and increasing in n , so

$$(J_X(2k, \gamma, B))^{1/2k} = \lim_{n \rightarrow \infty} \left\| \int_{|u| \leq n} |u|^\gamma \mu_B^*(u) du \right\|_{L^{2k}(dP)}.$$

where the term inside the above limit is increasing in n (because X is Gaussian). Hence, if $\{B_i, i \geq 1\}$ is a finite or countable collection of disjoint Borel sets in \mathbb{R}_+^2 , then since $\mu_B^*(u)$ is additive in B for each u , by Minkowski's inequality

$$(J_X(2k, \gamma, \bigcup_{i \geq 1} B_i))^{1/2k} \leq \sum_{i \geq 1} (J_X(2k, \gamma, B_i))^{1/2k}.$$

□

2.17 Estimation of J_X in special cases. We shall estimate the integral J_X , showing it to be finite, first in the case of Brownian motion and then in the case of what Geman, Horowitz and Rosen (1984) call "confluent Brownian motion" in \mathbb{R}^d given by the difference of two independent Brownian motions. The following lemma is required in both cases.

2.18 LEMMA. (Geman, Horowitz and Rosen 1984.) Let $(b(t), t \geq 0)$ be Brownian motion in \mathbb{R}^d . Given $\gamma \geq 0$ and $M \geq 0$, there exists a constant $c = c(d, M, \gamma)$ such that for all $k \in \mathbb{Z}_+$ and t_1, t_2, \dots, t_k with $0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq M$, we have

$$\int_{(\mathbb{R}^d)^k} \left(\prod_{\ell=1}^k |u_\ell|^\gamma \right) \exp\left\{-\frac{1}{2} \text{Var}\left(\sum_{j=1}^k u_j \cdot b(t_j)\right)\right\} d\bar{u} \leq c^k \prod_{j=1}^k (\Delta_j t)^{-(d/2)-\gamma} \quad (1)$$

where $\bar{u} = (u_1, \dots, u_k)$ and $\Delta_j(t) = t_j - t_{j-1}$ for each j (with $t_0 = 0$).

Proof. We have

$$\begin{aligned} \sum_{j=1}^k u_j \cdot b(t_j) &= \sum_{j=1}^k \sum_{i=1}^j u_j \cdot (b(t_i) - b(t_{i-1})) \\ &= \sum_{i=1}^k v_i \cdot (b(t_i) - b(t_{i-1})) \end{aligned} \quad (2)$$

where $v_i = \sum_{j=i}^k u_j$. Since Brownian motion has independent increments we have

$$\text{Var} \sum_{j=1}^k u_j \cdot b(t_j) = \sum_{i=1}^k |v_i|^2 \Delta_i t.$$

Also,

$$\begin{aligned} \prod_{\ell=1}^k |u_\ell|^\gamma &= \prod_{\ell=1}^k |v_\ell - v_{\ell+1}|^\gamma \\ &\leq 2^{k\gamma} \prod_{\ell=1}^k (|v_\ell| \vee |v_{\ell+1}|)^\gamma \\ &\leq 2^{k\gamma} \prod_{\ell=1}^k (|v_\ell|^{2\gamma} \vee 1). \end{aligned}$$

Changing variables from \bar{u} to $\bar{v} = (v_1, \dots, v_k)$, we see that the left hand side of (1) is at most

$$2^{k\gamma} \prod_{j=1}^k \left[\int_{\mathbb{R}^d} (|v_j|^{2\gamma} \vee 1) \exp\{-|v_j|^2 (\Delta_j t)/2\} dv_j \right]. \quad (3)$$

By a change of variables, the integral in (3) equals

$$\begin{aligned} & (\Delta_j t)^{-(d/2)-\gamma} \int_{\mathbb{R}^d} (|w|^{2\gamma} \vee (\Delta_j t)^\gamma) e^{-w^2/2} dw \\ & \leq (\Delta_j t)^{-(d/2)-\gamma} \int_{\mathbb{R}^d} (|w|^{2\gamma} \vee M^\gamma) e^{-w^2/2} dw. \end{aligned}$$

The w -integral is finite, so (1) follows. \square

2.19 Example of Theorem 2.14: Brownian motion on the line. We

shall now show that Theorem 2.14 is applicable when $X(t)$ is Brownian motion on the line. To use Theorem 2.14, we need to check that the Fourier transform of occupation measure is in $L^2(dx)$ almost surely, and that J_X is finite for suitable k and γ . This we do in the next two lemmas.

2.20 LEMMA. Let $N = d = 1$ and let $(X(t), t \geq 0)$ be Brownian motion on the line. Let $t_1 > 0$, and let $B = [0, t_1]$. Then the Fourier transform $\mu_B^*(\cdot)$ of the occupation measure of X relative to B is square-integrable almost surely.

Proof.

$$\begin{aligned} E \int_{-\infty}^{\infty} |\mu_B^*(u)|^2 du &= \int_{-\infty}^{\infty} E \left[\int_0^{t_1} \int_0^{t_1} \exp\{iu(X(t)-X(t'))\} dt dt' \right] du \\ &= \int_{-\infty}^{\infty} \int_0^{t_1} \int_0^{t_1} \exp\{-\frac{1}{2} u^2 |t-t'|\} dt dt' du. \end{aligned}$$

Taking the u -integral inside the t and t' -integrals, and changing variable to $u(|t-t'|)^{1/2}$, we have

$$E \int_{-\infty}^{\infty} |\mu_B^*(u)|^2 du \leq \text{const.} \cdot \int_0^{t_1} \int_0^{t_1} |t-t'|^{-1/2} dt dt' < \infty.$$

So the Fourier transform of the occupation measure of X relative to $[0, t_1]$ is square-integrable almost surely. \square

2.21 LEMMA. Let $N = d = 1$ and let $(X(t), t \geq 0)$ be Brownian motion on the line. Then given $M \geq 0$, for all $k \in \mathbb{Z}_+$, $\gamma \in [0, \frac{1}{2})$ and $B = [a_1, a_2] \subset [0, M]$,

$$J_X(k, \gamma, B) \leq \text{const.} \cdot k! (a_2 - a_1)^{k(\frac{1}{2} - \gamma)}$$

where the constant depends only on d , M and γ .

Proof. By definition,

$$J_X(k, \gamma, B) = \int_{B^k} \left[\int_{\mathbb{R}^k} \left(\prod_{\ell=1}^k |u_\ell|^\gamma \right) \exp\left\{-\frac{1}{2} \text{Var} \sum_{j=1}^k u_j X(t_j)\right\} d\bar{u} \right] d\bar{t} \quad (1)$$

where $\bar{u} = (u_1, u_2, \dots, u_k)$ and $\bar{t} = (t_1, \dots, t_k)$. For each permutation π of $\{1, 2, \dots, k\}$ let I_π be the expression in the right hand side of (1) with the region of the outer integral restricted to $\{\bar{t}: 0 \leq t_{\pi(1)} \leq t_{\pi(2)} \leq \dots \leq t_{\pi(k)}\}$. By taking the u_ℓ integrals in different orders for different π , we see that I_π is the same for all permutations π ; hence $J_X = k! I_\iota$, where ι is the identity permutation.

For $0 \leq t_1 \leq \dots \leq t_k$, by Lemma 2.18 the inner integral in the right hand side of (1) is at most $c^k \prod_{j=1}^k (\Delta_j t)^{-(\frac{1}{2}) - \gamma}$, where c depends only on M , d and γ . Hence, by changing variable in the outer integral from (t_1, \dots, t_k) to $(\Delta_1 t, \dots, \Delta_k t)$ we have for $0 \leq \gamma < \frac{1}{2}$,

$$I_\iota \leq c^k (\frac{1}{2} - \gamma)^{-k} (a_2 - a_1)^{k(\frac{1}{2} - \gamma)}.$$

Since $k!$ dominates $(c/(\frac{1}{2} - \gamma))^k$, the result follows. \square

Lemmas 2.20 and 2.21 (with k chosen suitably) together imply that Theorem 2.14 may be applied in the case of Brownian motion on the line. We shall see later that by using Kolmogorov's lemma in two parameters, we may by these methods recover the well-known result of Trotter (1958) that the local time of Brownian motion relative to $[0, t]$ can be taken to be jointly continuous in x and t .

2.22 Confluent Brownian motion. Let $b(t)$ and $\bar{b}(t)$ be independent Brownian motions in \mathbb{R}^d . Following Geman, Horowitz and Rosen (1984), we shall refer to the 2-parameter process $X(s, t)$ given by

$$X(s, t) = \bar{b}(t) - b(s) \quad (s, t) \in \mathbb{R}_+^2$$

as *Confluent Brownian motion* in \mathbb{R}^d . By definition, a local time of X is an intersection local time of two independent Brownian motions. For $d = 2$ or $d = 3$, we shall now show that Theorem 2.14 may be used to show that such a local time exists almost surely. First, we estimate J_X where X is confluent Brownian motion.

2.23 LEMMA. (essentially Geman, Horowitz and Rosen (1984)). Let $N = 2$ and let $d = 2$ or $d = 3$. Let $X(s, t)$ be confluent Brownian motion in \mathbb{R}^d . Let $M < \infty$. Then for $0 \leq \gamma < 2 - (d/2)$, $k \in \mathbb{Z}_+$ and all rectangles $B = [a_1, a_2] \times [a_3, a_4]$ in $[0, M] \times [0, M]$,

$$J_X(k, \gamma, B) \leq \text{const.} \cdot (k!)^2 (\lambda^2(B))^{k(1-(d/4)-(\gamma/2))}$$

Where the constant depends only on M , d and γ , and $\lambda^2(B)$ is the area of B , i.e. $\lambda^2(B) = (a_2 - a_1)(a_4 - a_3)$.

Proof. $X(s, t)$ is given by a difference of independent Brownian motions b and \bar{b} . Hence $J_X(k, \gamma, B)$ is equal to

$$\int_{a_1}^{a_2} \dots \int_{a_1}^{a_2} \int_{a_3}^{a_4} \dots \int_{a_3}^{a_4} \left\{ \int_{(\mathbb{R}^d)^k} \left(\prod_{\ell=1}^k |u_\ell|^\gamma \right) \exp \left\{ -\frac{1}{2} \left[\text{Var} \left(\sum_{j=1}^k u_j \cdot \bar{b}(t_j) \right) + \text{Var} \left(\sum_{i=1}^k u_i \cdot b(s_i) \right) \right] \right\} d\bar{u} \right\} dt_1 \dots dt_k ds_1 \dots ds_k.$$

By Cauchy-Schwarz, the innermost integral is at most

$$\left[\int_{(\mathbb{R}^d)^k} \left(\prod_{\ell=1}^k |u_\ell|^\gamma \right) \exp \left\{ -\text{Var} \sum_{i=1}^k u_i \cdot \bar{b}(t_i) \right\} d\bar{u} \right]^{\frac{1}{2}} \times \left[\int_{(\mathbb{R}^d)^k} \left(\prod_{\ell=1}^k |v_\ell|^\gamma \right) \exp \left\{ -\text{Var} \sum_{i=1}^k v_i \cdot b(s_i) \right\} d\bar{v} \right]^{\frac{1}{2}}.$$

Hence

$$J_X(k, \gamma, B) \leq I_{a_1, a_2} I_{a_3, a_4} \quad (1)$$

where for $0 \leq p \leq q$ we define

$$I_{p, q} = \int_p^q \dots \int_p^q \left[\int_{(\mathbb{R}^d)^k} \left(\prod_{\ell=1}^k |u_\ell|^\gamma \right) \exp \left\{ -\text{Var} \sum_{i=1}^k u_i \cdot \bar{b}(t_i) \right\} d\bar{u} \right]^{\frac{1}{2}} dt_1 \dots dt_k.$$

To estimate I_{a_1, a_2} note that, as in the proof of Lemma 2.21, if we split the region of the outer integral into $k!$ regions corresponding to permutations of $\{1, 2, \dots, k\}$, then each region makes the same contribution to the integral. Hence we need consider only the outer integral over

$\{(t_1, \dots, t_k) : a_1 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq a_2\}$. For such (t_1, \dots, t_k) , by Lemma 2.18 there exists $c = c(d, M, \gamma) > 0$ such that the inner integral is at most

$$c^k \prod_{i=1}^k (\Delta_i t)^{-(d/2) - \gamma}.$$

Hence, by changing variable from $(t_i, 1 \leq i \leq k)$ to $(\Delta_i t, 1 \leq i \leq k)$ we have

$$\begin{aligned}
I_{a_1, a_2} &\leq \text{const.} \cdot k! \left[\int_0^{a_2 - a_1} x^{-(d/4) - (\gamma/2)} dx \right]^k \\
&= \text{const.} \cdot k! \left[(a_2 - a_1)^{1 - (d/4) - (\gamma/2)} \right]^k
\end{aligned}$$

provided that $\gamma < 2 - (d/2)$. A similar expression for I_{a_3, a_4} , together with (1), gives us the desired result. \square

2.24 Remark. Suppose $b(t)$ is Brownian motion in \mathbb{R}^d ($d = 2$ or $d = 3$) and $X(s, t)$ is given by

$$X(s, t) = b(t) - b(s), \quad (s, t) \in \mathbb{R}_+^2,$$

i.e. X is a "self-confluent Brownian motion". Then if B is a neighbourhood of the diagonal $\{s = t\}$, $J_X(k, \gamma, B)$ is never finite (see Rosen (1983)).

Now, however, set B to be a bounded rectangle in the upper triangle $\{0 \leq s \leq t \leq \infty\}$ (possibly touching the diagonal). Then $\{X(s, t) : (s, t) \in B\}$ may be thought of as a confluent Brownian motion, by the markovian property of Brownian motion. Hence, for k and γ as in Lemma 2.23, $J_X(k, \gamma, B)$ satisfies the bound in Lemma 2.23.

Moreover, if instead B is an arbitrary bounded Borel subset of the upper triangle which is strictly separated from the diagonal, then B is contained in a finite union of rectangles in the upper triangle, so $J_X(k, \gamma, B)$ is finite. We also have the following:

2.25 LEMMA. (Rosen (1983)). Let $d = 2$ or $d = 3$. Let $X(s,t)$ be as in Remark 2.24. Then for all bounded Borel B in \mathbb{R}_+^2 , the occupation measure of X relative to B has a square-integrable Fourier transform, almost surely.

Proof. This is straightforward integration: see the proof of Theorem 1 of Rosen (1983).

2.26 Remarks. It follows from Lemma 2.25 that if $X(s,t)$ is a confluent Brownian motion in \mathbb{R}^d ($d=2$ or $d=3$), for all bounded Borel B in \mathbb{R}_+^2 the occupation measure relative to B has square-integrable Fourier transform. This, together with Lemma 2.23, means that we can apply Theorem 2.14; hence, confluent Brownian motion in \mathbb{R}^d has a (continuous) local time relative to B almost surely. In other words, two Brownian motions have a continuous intersection local time relative to B , almost surely.

Similarly, Brownian motion in \mathbb{R}^d has a continuous self-intersection local time relative to B almost surely, if B is a rectangle in the upper triangle or B is strictly separated from the diagonal.

2.27 Remark. In addition to the above, Geman, Horowitz and Rosen (1984) consider confluences of more than two Brownian paths in \mathbb{R}^2 via the process

$$X(t_0, t_1, \dots, t_N) = \left[b_1(t_1) - b_0(t_0), b_2(t_2) - b_1(t_1), \dots, b_N(t_N) - b_{N-1}(t_{N-1}) \right] \\ \in \mathbb{R}^{2N},$$

where b_0, b_1, \dots, b_N are independent Brownian motions in \mathbb{R}^2 , and show that $J_X(2k, \gamma, B) < \infty$ for all $k \in \mathbb{Z}_+$ and $\gamma < 1/N$. It is

possible to extend the methods of the next chapter to proofs of the existence of local time for quasi-every such X .

CHAPTER 3 THE EXISTENCE OF LOCAL TIMES FOR QUASI-EVERY BROWNIAN PATH

Let $(b^\tau(t), t \geq 0, \tau \geq 0)$ be a Brownian sheet taking values in \mathbb{R} (resp. \mathbb{R}^d , $d=2$ or $d=3$). In this chapter we show that with probability one, the path $b^\tau(\cdot)$ has a local time (resp. a self-intersection local time) for all $\tau \geq 1$ simultaneously. Since the Ornstein-Uhlenbeck process on Wiener space is a smooth re-normalisation of the Brownian sheet, this is equivalent to saying that quasi-every Brownian path $b(t)$ in \mathbb{R} (resp. \mathbb{R}^d , $d=2$ or $d=3$) has a local time (resp. a self-intersection local time).

3.1 Definitions. We first consider the self-intersection local time, relative to subsets of the bounded triangular set Δ defined by

$$\Delta = \{(s, t) : 0 \leq s < t \leq 1\}$$

(the choice of the upper bound on t here is arbitrary). Let $(b^\tau(t), \tau \geq 0, t \geq 0)$ be a two-parameter Wiener process in \mathbb{R}^d , where $d=2$ or $d=3$. For $(s, t) \in \mathbb{R}_+^2$, set

$$X^\tau(s, t) = b^\tau(t) - b^\tau(s)$$

and (setting $\tau = 1$)

$$X(s, t) = X^1(s, t).$$

We now show that quasi-every Brownian path has a continuous self-intersection local time in the sense of Rosen (1983). Denote the class of Borel sets in Δ by $\mathcal{B}(\Delta)$, and the rectangle with opposite corners at (s_1, t_1) and (s_2, t_2) by $R(s_1, t_1, s_2, t_2)$ (all rectangles here have sides parallel to the axes).

3.2 THEOREM. There exists a random function

$$\varphi(\tau, x, B) = \varphi(\omega, \tau, x, B) \quad (\tau \geq 1, x \in \mathbb{R}^d, B \in \mathcal{B}(\Delta))$$

such that the following hold almost surely:

(i) For each $B \in \mathcal{B}(\Delta)$ and $\tau \geq 1$, $\varphi(\tau, \cdot, B)$ is a (not necessarily continuous) version of the self-intersection local time in B of the path $b^\tau(\cdot)$ (i.e. the local time in B of X^τ).

(ii) For each $\tau \geq 1$ and $x \in \mathbb{R}^d$, $\varphi(\tau, x, \cdot)$ is a σ -finite measure on the Borel subsets of Δ .

(iii) The function h defined by

$$h(\tau, x, s_1, t_1, s_2, t_2) = \varphi(\tau, x, R(s_1, t_1, s_2, t_2))$$

on $\{(\tau, x, s_1, t_1, s_2, t_2) : \tau \geq 1, x \in \mathbb{R}^d, R(s_1, t_1, s_2, t_2) \subset \Delta\}$

is Hölder continuous of any order less than $1-d/4$.

In particular, if $B \subset \Delta$ is a rectangle (possibly touching the diagonal), $\varphi(\tau, \cdot, B)$ is continuous for all τ .

3.3 Remarks. Theorem 3.2 shows that the statements of Theorems 1 and 3 of Rosen (1983) hold quasi-everywhere. We shall see later that the Hölder continuity of h in the x , s_i and t_i directions is stronger than that given by property (iii).

3.4 Proof of Theorem 3.2. Write $(\overline{s}, \overline{t})$ for $(s_1, t_1, \dots, s_{2k}, t_{2k})$, and \overline{u} for (u_1, \dots, u_{2k}) . For $k \in \mathbb{Z}_+$, $\gamma > 0$ and $B \subset \Delta$, by definition (see section 2.13), $J_{X^\tau}(2k, \gamma, B)$ is equal to

$$\int_{\mathbb{R}^{2kd}} \left\{ \int_B \left(\prod_{\ell=1}^{2k} |u_\ell|^\gamma \right) \exp\left\{-\left(\frac{1}{2}\right) \text{Var}\left(\sum_{j=1}^{2k} u_j \cdot X^\tau(s_j, t_j)\right)\right\} d(\overline{s}, \overline{t}) \right\} d\overline{u}.$$

$X^\tau(\cdot)$ has the same law as $\tau^{\frac{1}{2}} X^1(\cdot) = \tau^{\frac{1}{2}} X(\cdot)$ by the scaling

property for the Brownian sheet (section 1.5). Hence by the change of variable $u'_j = \tau^{\frac{1}{2}} u_j$,

$$J_{X^\tau}(k, \gamma, B) \leq J_X(k, \gamma, B) \quad (\tau \geq 1). \quad (1)$$

Let $\mathcal{R}(\Delta)$ denote the set of rectangles in Δ (possibly touching the diagonal). Given B in $\mathcal{R}(\Delta)$ and $\tau \geq 1$, $J_X(2k, 0, B)$ is finite ($k \in \mathbb{Z}_+$) by Lemma 2.23. Hence, so is $J_{X^\tau}(2k, 0, B)$.

Also, by Lemma 2.25 the Fourier transform of the occupation measure of X^τ relative to B is square integrable almost surely, so Lemma 2.10 is applicable.

For $x \in \mathbb{R}^d$, $\tau \geq 1$ and $B \in \mathcal{R}(\Delta)$, define the random variable $\varphi(\tau, x, B)$ by the improper integral

$$\varphi(\tau, x, B) = \int_{u \in \mathbb{R}^d} \int_{(s, t) \in B} \exp\{iu \cdot (X^\tau(s, t) - x)\} ds dt du \quad (2)$$

in the sense of Lemma 2.10. That is to say, for $k \in \mathbb{Z}_+$, $\varphi(x, \tau, B)$ is an $L^{2k}(dP)$ limit of real-valued random variables given by restricting the integral in (2) to $\{|u| \leq m\}$. It follows that in taking the $2k^{\text{th}}$ moment of φ we may take the expectation inside the multiple improper integral; we shall do this below without comment.

Define the distance between two rectangles to be the maximum distance between corresponding corners. We shall use Kolmogorov's lemma to show that $\varphi(\tau, x, B)$ ($\tau \geq 1$, $x \in \mathbb{R}^d$, $B \in \mathcal{R}(\Delta)$) can be modified to be jointly continuous in x , τ and B . Fix $k \in \mathbb{Z}_+$. We make the following estimates of $(2k)^{\text{th}}$ moments.

Firstly, we follow Rosen (1983). For x and y in \mathbb{R}^d , $\tau \geq 1$, and $B \in \mathcal{B}(\Delta)$, since φ is real

$$\begin{aligned} & E |\varphi(\tau, x, B) - \varphi(\tau, y, B)|^{2k} = E [\varphi(\tau, x, B) - \varphi(\tau, y, B)]^{2k} \\ & = E \left[\int_{u \in \mathbb{R}^d} \int_{(s, t) \in B} [\exp(-iu \cdot x) - \exp(-iu \cdot y)] \exp\{iu \cdot X^\tau(s, t)\} ds dt du \right]^{2k} \\ & = \int_{(\mathbb{R}^d)^{2k}} \left\{ \int_B \left[\prod_{\ell=1}^{2k} \{\exp(-iu_\ell \cdot x) - \exp(-iu_\ell \cdot y)\} \right] E \exp\{i \sum_{j=1}^{2k} u_j \cdot X^\tau(s_j, t_j)\} \right. \\ & \quad \left. d(\overline{s}, \overline{t}) \right\} d\overline{u}. \end{aligned}$$

For $0 \leq \gamma < 1$,

$$|\exp(-iu_\ell \cdot x) - \exp(-iu_\ell \cdot y)| \leq 2|u_\ell|^\gamma |y - x|^\gamma,$$

so by definition

$$\begin{aligned} E |\varphi(\tau, x, B) - \varphi(\tau, y, B)|^{2k} & \leq 2^{2k} |y - x|^{2k\gamma} J_{X^\tau}(2k, \gamma, B). \\ & \leq 2^{2k} |y - x|^{2k\gamma} J_X(2k, \gamma, B). \end{aligned} \quad (*)$$

Secondly, for $y \in \mathbb{R}^d$, $\tau > \sigma \geq 1$, and $B \in \mathcal{B}(\Delta)$,

$$\begin{aligned} E |\varphi(\tau, y, B) - \varphi(\sigma, y, B)|^{2k} & = \int_{\overline{u} \in \mathbb{R}^{2dk}} \int_{(\overline{s}, \overline{t}) \in B^{2k}} \left\{ \left(\prod_{\ell=1}^{2k} \exp(-iu_\ell \cdot y) \right) \right. \\ & \quad \times E \prod_{j=1}^{2k} [\exp\{iu_j \cdot X^\tau(s_j, t_j)\} - \exp\{iu_j \cdot X^\sigma(s_j, t_j)\}] \left. \right\} d(\overline{s}, \overline{t}) d\overline{u}. \end{aligned} \quad (3)$$

The first term in the integrand in (3) has unit modulus, while the second term is equal to the expectation of Z , defined by

$$Z = \left[\prod_{j=1}^{2k} \exp\{iu_j \cdot X^\sigma(s_j, t_j)\} \right] \left[\prod_{\ell=1}^{2k} (\exp\{iu_\ell \cdot (X^\tau(s_\ell, t_\ell) - X^\sigma(s_\ell, t_\ell))\} - 1) \right].$$

The two factors in Z are independent of one another, and the

expectation of the first factor is $\exp\{-\frac{1}{2} \text{Var}(\sum_{j=1}^{2k} u_j \cdot X^\sigma(s_j, t_j))\}$.

The modulus of the second factor in Z is at most 2^{2k} times

$$\prod_{\ell=1}^{2k} |u_{\ell}|^{\gamma} |X^{\tau}(s_{\ell}, t_{\ell}) - X^{\sigma}(s_{\ell}, t_{\ell})|^{\gamma}.$$

Now $(\tau - \sigma)^{1/2}(b^{\tau}(t) - b^{\sigma}(t))_{t \geq 0}$ is a Brownian motion, so

$$\sup \{E \prod_{\ell=1}^{2k} |X^{\tau}(s_{\ell}, t_{\ell}) - X^{\sigma}(s_{\ell}, t_{\ell})|^{\gamma}\} = (\tau - \sigma)^{k\gamma} \sup \{E \prod_{\ell=1}^{2k} |X(s_{\ell}, t_{\ell})|^{\gamma}\} \quad (4)$$

where the supremum on each side of (4) is over

$\{(\overline{s}, \overline{t}) : (s_{\ell}, t_{\ell}) \in \Delta, 1 \leq \ell \leq 2k\}$. Hence the modulus of the expectation of the second factor in Z is bounded by a constant multiple of

$(\tau - \sigma)^{k\gamma} \prod_{\ell=1}^{2k} |u_{\ell}|^{\gamma}$. Putting together these estimates for the integrand

in (3), and applying (1), we have

$$E|\varphi(\tau, y, B) - \varphi(\sigma, y, B)|^{2k} \leq c J_X(2k, \gamma, B) \cdot |\tau - \sigma|^{k\gamma} \quad (**)$$

$$(y \in \mathbb{R}^d, \tau > \sigma \geq 1)$$

where c depends only on k and γ .

Thirdly, suppose B_1 and B_2 are in $\mathcal{R}(\Delta)$. Then

$\varphi(\sigma, y, B_2) - \varphi(\sigma, y, B_1)$ is the sum of at most four terms of the form $\pm \varphi(\sigma, y, B)$, where $B \in \mathcal{R}(\Delta)$ and $\lambda^2(B) \leq \text{dist.}(B_1, B_2)$. But for $y \in \mathbb{R}^d$, $\sigma \geq 1$, and $B \in \mathcal{R}(\Delta)$,

$$\begin{aligned}
E(\varphi(\sigma, y, B))^{2k} &= \int_{\mathbb{R}^{2dk}} \int_B \left[\prod_{\ell=1}^{2k} \exp(-iu_\ell \cdot y) \right] \cdot E \exp(i \sum_{j=1}^{2k} u_j \cdot X^\sigma(s_j, t_j)) d(\overline{s, t}) d\bar{u} \\
&\leq J_X(2k, 0, B) \\
&\leq c_0^{2k} ((2k)!)^2 (\lambda^2(B))^{(1-(d/4))2k} \quad (***)
\end{aligned}$$

by Lemma 2.23. Here c_0 depends only on d and τ_1 .

By the above three estimates (*), (**) and (***), combined with Minkowski's inequality in $L^{2k}(\text{Prob.})$, we find that if $0 \leq \gamma < 2-d/2$, then for $\tau \geq \sigma \geq 1$, x and y in \mathbb{R}^d and rectangles $P = R(s_1, t_1, s_2, t_2)$ and $P' = R(s'_1, t'_1, s'_2, t'_2)$ in $\mathcal{R}(\Delta)$,

$$\begin{aligned}
E|\varphi(\tau, y, P') - \varphi(\sigma, x, P)|^{2k} &< c (|y-x|^\gamma + |\tau-\sigma|^{\gamma/2} + (\text{dist.}(P, P'))^{1-d/4})^{2k} \\
&\leq c' |(\tau, y, s'_1, t'_1, s'_2, t'_2) - (\sigma, x, s_1, s_2, t_1, t_2)|^{\gamma k}.
\end{aligned}$$

Here the constant c' depends only on k and γ .

Take a modification of φ (also denoted φ) so that h given by $h(\tau, x, s_1, s_2, t_1, t_2) \doteq \varphi(\tau, x, R(s_1, s_2, t_1, t_2))$ is a separable process (see Doob (1953), Theorem 2.4). By Kolmogorov's lemma (see Meyer (1981), Garsia (1971)) h is almost surely Hölder continuous (on the domain of interest in \mathbb{R}^{5+d}) of any order less than $(k\gamma - 5 - d)/2k$, and hence (by allowing $k \rightarrow \infty$ and $\gamma \rightarrow 2-d/2$), of any order less than $1-d/4$, so property (iii) in the statement of the theorem holds.

We now extend the definition of $\varphi(\tau, x, B)$ to all Borel sets B in Δ . Let $\mathcal{Q}(\Delta)$ denote those rectangles in Δ with rational corners. The following argument holds for almost all ω . For all $R \in \mathcal{Q}(\Delta)$ and rational τ , $\varphi(\tau, \cdot, R)$ is a version of the local time

of X^τ in R by Lemma 2.10. Hence $\varphi(\tau, x, \cdot)$ is finitely additive on $\mathcal{Q}(\Delta)$ for almost all x , and hence for all (τ, x) by joint continuity in (τ, x) . Fix x and τ . $\varphi(\tau, x, \cdot)$ is a finitely additive, continuous function on rectangles in $\mathcal{Q}(\Delta)$. By classical measure theory, φ extends to a σ -finite measure on the Borel sets in Δ (which is a countable union of rectangles). We denote this measure also by $\varphi(\tau, x, \cdot)$. So property (ii) in the statement of the theorem holds.

It remains to verify that property (i) holds, i.e. that for all τ and B $\varphi(\tau, \cdot, B)$ is a version of the local time of X^τ in B . This holds for $B \in \mathcal{Q}(\Delta)$ and rational τ by the definition of φ together with Lemma 2.12. So if $f \in C_0(\mathbb{R}^d)$ we have for $B \in \mathcal{Q}(\Delta)$ and rational τ

$$\int_B f(X^\tau(s, t)) ds dt = \int_{\mathbb{R}^d} f(x) \varphi(\tau, x, B) dx. \quad (5)$$

Following Shigekawa (1984), we deduce from continuity in τ that (5) holds for all $\tau \geq 1$ and $B \in \mathcal{Q}(\Delta)$. The set of $B \subset \Delta$ for which (5) holds for all τ is a monotone class, so (5) holds for all $\tau \geq 1$ and all Borel B in Δ . This completes the proof. \square

3.5 Remark. For fixed τ , $\varphi(\tau, \cdot, \Delta)$ is discontinuous at the origin (see Rosen (1983)). On the other hand, the proof of Theorem 3.1 uses the finiteness of $J_X(2k, \gamma, B)$ for rectangular B in Δ (possibly touching the diagonal) to show that $\varphi(\tau, x, B)$ is continuous in τ and x for such B . This suggests the following:

3.6 THEOREM. Let B_0 be a fixed Borel set in Δ , such that (with $X(s,t)$ as above) $J_X(2k, \gamma, B_0)$ is finite for any $k \in \mathbb{Z}_+$ and any $\gamma \in [0, 2-d/2)$. Then there exists a random function

$$\phi(\tau, x) = \phi_{B_0}(\omega, \tau, x) \quad (\tau \geq 1, x \in \mathbb{R}^d)$$

such that we have almost surely

- (i) $\phi(\tau, \cdot)$ is the local time of $X^\tau(\cdot)$ relative to B_0 ($\tau \geq 1$).
- (ii) $\phi(\tau, x)$ is jointly Holder continuous in τ and x of any order less than $1-d/4$.

Proof. The proof is essentially the same as that of Theorem 3.1, but easier since we need consider only the single set B_0 rather than a collection of rectangles. □

3.7 Remark. By Lemma 2.23 and Remark 2.24, we know that B_0 satisfies the above hypothesis if it is a rectangle (possibly touching the diagonal) or if it is strictly separated from the diagonal.

A similar method yields an alternative to the original proof by Shigekawa (1984) of the existence of a local time for quasi-every Brownian path on the line:

3.8 THEOREM. Let $(X^\tau(t), \tau \geq 0, t \geq 0)$ be a Brownian sheet in \mathbb{R} . There exists almost surely a random, real-valued function $\alpha(\tau, x, t)$ defined on $\tau > 0, x \in \mathbb{R}, t \geq 0$, such that

(i) For all τ and t , $\alpha(\tau, \cdot, t)$ is the local time of the function $X^\tau(\cdot)$ relative to the set $[0, t]$.

(ii) $\alpha(\tau, x, t)$ is jointly Hölder continuous in (τ, x, t) , of any order less than $1/4$.

3.9 Remarks. Shigekawa (1984) proved the existence of $\alpha(\tau, t, x)$ satisfying condition (i) and jointly continuous in t, τ and x , using Tanaka's formula for the local time of Brownian motion on the line. Our proof uses instead the Fourier-analytic expression for the local time described in Chapter 2. Our method has the advantages of giving us a stronger modulus of continuity in τ for the local time, and of being easily extended to the case where τ is multidimensional (i.e. $\tau \in \mathbb{R}_+^N$ so that $(X^\tau(t), t \in \mathbb{R}_+, \tau \in \mathbb{R}_+^N)$ is an $(N+1)$ -parameter Wiener process).

3.10 Remarks on the proof of Theorem 3.6. The proof is virtually the same as that of Theorem 3.2. For fixed τ and t , $\alpha(\tau, x, t)$ is defined by

$$\alpha(\tau, x, t) = \int_{u=-\infty}^{\infty} \int_{s=0}^t \exp(iu(X^\tau(s) - x)) \, ds \, du .$$

in the sense of Lemma 2.10. Lemma 2.10 is applicable because by Lemma 2.20, the Fourier transform of the occupation measure of X^τ is square-integrable almost surely, while by Lemma 2.21, if

$0 \leq \gamma < \frac{1}{2}$, the integral $J_{X^\tau}(k, \gamma, [0, t])$ is finite for all $k \in \mathbb{Z}_+$.

Here we have used the scaling property relating X^τ to X^1 (section 1.5). By estimates similar to (*), (**) and (***) in the proof of Theorem 3, we find that for $t_0 > 0$, $\gamma < \frac{1}{2}$ and $k \in \mathbb{Z}_+$,

$$\mathbb{E} |\alpha(\tau, x, t) - \alpha(\sigma, y, s)|^{2k} \leq \text{const.} \cdot (|\tau - \sigma|^{k\gamma} + |x - y|^{2k\gamma} + |t - s|^k) \quad (****)$$

where the constant depends only on t_0 , γ and k (For the estimate of $\mathbb{E} |\alpha(\tau, x, t) - \alpha(\tau, x, s)|^{2k}$ we have used the case $\gamma = 0$ of Lemma 2.21). Hence by Kolmogorov's lemma, α has a modification (also denoted α) which is Hölder continuous of any order less than $(k\gamma - 3)/2k$, and hence of any order less than $1/4$. As in the proof of Theorem 3.2, α is the desired local time for all τ , x and t .

In this chapter we show that certain Hölder-type conditions on the local times of a Brownian path hold for quasi-every path. Rosen (1983) gave global Hölder conditions on the self-intersection local time of almost every Brownian path in \mathbb{R}^d ($d=2$ or $d=3$); here we show that these conditions hold for quasi-every path. We also give analogous quasi-everywhere results for the local time of a Brownian path on the line.

All the proofs here use the same idea, which is to approximate an increment in the local time for arbitrary τ in an interval by an increment in the local time for τ on a lattice just fine enough to make the approximation a good one.

4.1 Hölder continuity and the lemmas of Kolmogorov and Garsia. In Chapter 3 we used Kolmogorov's lemma to prove joint continuity of local times in the space parameter, the "ordinary" time parameter t and the time parameter τ of the Ornstein-Uhlenbeck process in Wiener space. It is well known that the proof of Kolmogorov's lemma (see Meyer (1981) or Garsia (1971)) implies that a separable random function $X(\underline{t})$ defined on \mathbb{R}^N and satisfying a suitable uniform condition on the moments of its increments $|X(\underline{t}) - X(\underline{s})|$ in terms of $|\underline{t} - \underline{s}|$ is not only continuous but satisfies a uniform Hölder condition in \underline{t} almost surely.

We shall now strengthen this result by showing that when the moment conditions are not homogeneous, the random function $X(\underline{t})$ satisfies different Hölder conditions in different directions. Such

results are needed if we wish to find the best possible Hölder conditions which hold for the local times of quasi-every path. This means that we want Hölder conditions holding for each τ on local times, considered as functions of x or t .

4.2 LEMMA. Let $X(t)$ be a separable (or continuous) \mathbb{R}^d -valued process with parameter $t = (t_1, t_2, \dots, t_N)$ taking values in an open subset D of \mathbb{R}^N . Suppose there exist constants $\alpha_1, \alpha_2, \dots, \alpha_N$, with $N < \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_N$, such that for any compact $K \subset D$ there exists a constant $c > 0$ such that

$$E |X(t) - X(s)|^r \leq c \sum_{i=1}^N |t_i - s_i|^{\alpha_i} \quad (\text{all } t \text{ and } s \text{ in } K)$$

(without loss of generality $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_N$).

Then $X(t)$ is continuous in t . Also, if γ_i ($1 \leq i \leq N$) are chosen so that

$$0 < \gamma_1 < \alpha_1 - N$$

$$0 < \gamma_j(1 + \gamma_1^{-1} + \dots + \gamma_{j-1}^{-1}) < \alpha_j - (N - (j-1)) \quad (2 \leq j \leq N),$$

then for all compact $K \subset D$ there exists a finite random variable $C = C(\omega)$ such that for all s and t in K ,

$$|X(t) - X(s)| \leq C \left[\sum_{i=1}^d |t_i - s_i|^{\gamma_i/r} \right].$$

4.3 Remarks. The first part of this result is Kolmogorov's lemma in N parameters. The second part says that X is Hölder continuous of order γ_j in the j^{th} co-ordinate ($1 \leq j \leq N$). Our condition on the $(\gamma_j, 1 \leq j \leq N)$ is certainly attained if we take $0 < \gamma < \alpha_1 - d$, and set $\gamma_j = \gamma$ for all j . Thus we recover the result obtained directly from the Kolmogorov-Garsia lemma, that $X(t)$ is

jointly Hölder continuous in (t_1, \dots, t_N) of any order less than $(\alpha - N)/r$.

We shall be interested in cases where it is possible to estimate r^{th} moments for arbitrarily large r (e.g. when X is a Gaussian process). Thus we may use the following immediate consequence of Lemma 4.2, which gives us a more manageable condition for the moduli of continuity;

4.4 COROLLARY. Suppose that for some positive β_i ($1 \leq i \leq N$) and some sequence r_n such that $r_n \rightarrow \infty$, X satisfies the following condition for all n and compact K in D :

$$E |X(t) - X(s)|^{r_n} \leq c(K, r_n) \left[\sum_{i=1}^N |t_i - s_i|^{\beta_i r_n} \right] \quad \text{all } s, t \text{ in } D.$$

Then X is Hölder continuous in the i^{th} co-ordinate t_i of any order less than β_i .

4.5 Proof of Lemma 4.2. Let K be a compact subset of D . Let $\epsilon > 0$ be such that $K_{2\epsilon} \subset D$, where $K_{2\epsilon}$ is the 2ϵ -neighbourhood of K . Let e_i be the unit vector in the direction of the i^{th} co-ordinate. We shall prove by induction on j that

$$\sup \{ |X(t + he_i) - X(t)| / |h|^{\gamma_i/r} : t \in K, 0 \leq h \leq \epsilon \} < \infty \quad (1 \leq i < j)$$

and

$$\sup \{ |X(t + he_j) - X(t)| / |h|^{\gamma_j/r} : t \in K, 0 \leq h \leq \epsilon \} < \infty \quad (j \leq i \leq N).$$

When the induction reaches $j = N$, the proof is complete.

The inductive step is as follows.



Let E_n be the lattice of points $t = (t_1, \dots, t_N)$ in D such that

$$\begin{aligned} t_i &\in 2^{-n}Z & j \leq i \leq N \\ t_j &\in 2^{-n\gamma_j/\gamma_i}Z & 1 \leq i < j \end{aligned}$$

Then

$$\#(E_n \cap K_\epsilon) = O(2^{n\zeta})$$

Where

$$\zeta = (N-j+1) + \sum_{i=1}^{j-1} \gamma_j/\gamma_i$$

$$< \alpha_j - \gamma_j \text{ by our condition on } \gamma_j.$$

By Chebyshev's inequality and our hypothesis, for $i \geq j$ and $t \in K$

$$\begin{aligned} P[|X(t+2^{-n}e_i) - X(t)| > (2^{-n})^{\gamma_j/r}] &\leq 2^{-n\gamma_j} E[|X(t+2^{-n}e_i) - X(t)|^r] \\ &\leq 2^{-n(\gamma_j - \alpha_i)} \leq 2^{-n(\gamma_j - \alpha_j)}. \end{aligned}$$

Hence for $j \leq i \leq N$,

$$P\left[\bigcup_{t \in E_n \cap K_\epsilon} \{ |X(t+2^{-n}e_i) - X(t)| > 2^{-n\gamma_j/r} \} \right] \leq (2^n)^{\gamma_j - \alpha_j + \zeta}.$$

By the Borel-Cantelli lemma there exists almost surely n_0 such that

$2^{-n_0} < \epsilon$ and for $n \geq n_0$, $t \in E_n \cap K_\epsilon$ and $j \leq i \leq N$,

$$|X(t+2^{-n}e_i) - X(t)| < 2^{-n\gamma_j/r} \quad (t \in E_n \cap K_\epsilon). \quad (1)$$

Let $n \geq n_0$ and let $t = (t_1, \dots, t_N) \in K$ be such that $2^n t_i \in Z$,

$j \leq i \leq N$. We can take $t' = (t'_1, \dots, t'_N)$ in $E_n \cap K_\epsilon$ such that

$$|t_i - t'_i| \leq 2^{-n\gamma_j/\gamma_i} \quad (1 \leq i < j). \text{ The Hölder continuity in the first}$$

$(j-1)$ co-ordinates (the inductive hypothesis) gives us

$$|X(t) - X(t')| \leq C' 2^{-n\gamma_j/r}$$

where C' is a global (random) constant. Hence for $j \leq i \leq N$,

$$\begin{aligned} |X(t+2^{-n}e_i) - X(t)| &\leq |X(t) - X(t')| + |X(t'+2^{-n}e_i) - X(t')| + \\ &\quad + |X(t+2^{-n}e_i) - X(t'+2^{-n}e_j)| \end{aligned}$$

$$\leq (2C'+1) 2^{-n(\gamma_i/r)}.$$

Now let $t \in K$ be arbitrary. By using the binary expansion of the i^{th} co-ordinate for each $i \geq j$, and summing a geometric series in a standard way (see McKean (1969), page 16), we can find C such that

$$|X(t+he_i) - X(t)| \leq C h^{\gamma_i/r}, \quad h < \epsilon, \quad t \in K, \quad j \leq i \leq N.$$

The induction is complete. \square

4.6 Quasi-everywhere Hölder conditions on self-intersection local

time. Let d be 2 or 3. As in the last chapter, let $(b^\tau(t), (t, \tau) \in \mathbb{R}_+^2)$ be a Brownian sheet in \mathbb{R}^d and let B_0 be a subset of the triangular set $\Delta \subset \mathbb{R}_+^2$, such that $J_X(2k, \gamma, B)$ is finite for $k \in \mathbb{Z}_+$ and $\gamma < 2-d/2$, where

$$X(s, t) = b^1(t) - b^1(s) \quad (s, t) \in \mathbb{R}_+^2.$$

Let $\varphi(\tau, x, B) \quad (\tau \geq 1, x \in \mathbb{R}^d, B \subset B_0)$

and $\phi(\tau, x) \quad (\tau \geq 1, x \in \mathbb{R}^d)$

be as in Theorems 3.2 and 3.6 respectively. That is, $\varphi(\tau, \cdot, B)$ (resp. $\phi(\tau, \cdot)$) is the continuous version of the self-intersection local time in B (resp. B_0) of $X^\tau(\cdot)$.

Here we give Hölder conditions holding for all τ , on $\varphi(\tau, x, B)$ as a function of the space variable x (Theorem 4.7) or the set variable B (Theorem 4.8). These results are stronger than those which would be obtained by direct application of the joint Hölder continuity of φ in x , τ and the corners of B . This is

because the Hölder continuity in the τ direction is weaker than that in the x or B directions.

4.7 THEOREM. The function $\phi(\tau, \cdot)$ is Hölder continuous of any order less than $2-d/2$ for all $\tau \geq 1$, almost surely.

4.8 THEOREM. Given $\tau_1 \geq 1$, there is a finite constant C and a finite random variable δ such that with probability 1, for every square $B \subset \Delta$ of the form

$$B = (p2^{-n}, (p+1)2^{-n}) \times (q2^{-n}, (q+1)2^{-n})$$

where p and q are integers and $2^{-n} < \delta$ we have

$$\phi(\tau, x, B) < C (\lambda^2(B))^{1-d/4} |\log \lambda^2(B)|^2 \quad (\text{all } x \in \mathbb{R}^d, \tau \in [1, \tau_1]).$$

4.9 COROLLARY. Let B_1 be a compact set in Δ and let $\tau_1 \geq 1$.

Then there exists a finite constant C and a positive random variable δ such that with probability 1, for every square B of the form $B = (a, a+h) \times (b, b+h) \subset B_1$ with $h < \delta$ we have

$$\phi(\tau, x, B) < C (\lambda^2(B))^{1-d/4} |\log \lambda^2(B)|^2 \quad (x \in \mathbb{R}^d, \tau \in [1, \tau_1]).$$

4.10 Remark. This last result is a global Hölder condition in the set variable B , and gives us an upper bound on the difference in ϕ between two sets in terms of the Hausdorff measure of their symmetric difference (see Lemma 2.5). The result implies that Theorem 3 of Geman, Horowitz and Rosen (1984) holds quasi-everywhere, with the upper bound changing only by a constant. However, our methods do not give us any quasi-everywhere local Hölder conditions on B such as in Theorem 2 of Geman, Horowitz and Rosen (1984) or Theorem 5 of Rosen (1983).

4.11 Proof of Theorem 4.7. Corollary 4.4 is designed for

application here. To be able to apply it, we need only refer to our estimates (*) and (**) for the $2k^{\text{th}}$ moments of $|\phi(\tau, x) - \phi(\tau, y)|$ and of $|\phi(\tau, x) - \phi(\sigma, x)|$, from section 3.4 (which apply equally well to ϕ as to φ). Corollary 4.4 does the rest. \square

4.12 Proof of Theorem 4.8. Our proof roughly follows that of

Theorem 3 of Geman, Horowitz and Rosen (1984). In the proof of Theorem 3.2 (section 3.4), we obtained a uniform estimate (***) for the $2k^{\text{th}}$ moment of $\varphi(\sigma, s, B)$ for rectangular B . This leads to a uniform estimate for $E \exp(\zeta \alpha(\tau, x, B) / (\lambda^2(B))^{1-d/4})$ for rectangular B and suitable choice of ζ ; by Chebyshev's inequality we may find positive constants c and c' such that

$$P(\varphi(\tau, x, B) \geq z^2 (\lambda^2(B))^{1-d/4}) \leq c e^{-c'z} \quad (1)$$

for all rectangles $B \subset \Delta$, all $\tau \in [1, t_1]$, $x \in \mathbb{R}^d$ and $z \geq 0$ (see Lemma (3.14) of Geman, Horowitz and Rosen (1984) for details).

Let Z^d be the integer lattice in \mathbb{R}^d . Let

$$D_n = \{x \in 2^{-n} Z^d: |x| \leq n\}, \quad E_n = \{\tau \in 2^{-3n} Z: 1 \leq \tau \leq \tau_1\}.$$

Let \mathcal{S}_n be the collection of all squares B in B_0 of the form

$$B = (i2^{-n}, (i+1)2^{-n}) \times (j2^{-n}, (j+1)2^{-n}).$$

Then the cardinalities of \mathcal{S}_n satisfy $\#(\mathcal{S}_n) = O(2^{2n})$ as $n \rightarrow \infty$.

For $k \in \mathbb{Z}_+$, $\tau \in [1, \tau_1]$ and $B \in \mathcal{S}_n$, we have by (1)

$$\begin{aligned} P[\varphi(\tau, x, B) \geq C (\lambda^2(B))^{1-d/4} |\log \lambda^2(B)|^2] &\leq c \exp\{-c'C |\log \lambda^2(B)|\} \\ &\leq c 2^{-2c'Cn}. \end{aligned}$$

Hence

$$P[\varphi(\tau, x, B) \geq C (\lambda_2(B))^{1-d/4} |\log \lambda^2(B)|^2, \text{ some } B \in \mathcal{S}_n, \tau \in E_n, x \in D_{3n}] \\ \leq \text{const. } n^d 2^{(5+3d-2c'C)n},$$

so provided C is chosen large enough, we may apply Borel-Cantelli.

Then there exists almost surely $n_0 \in \mathbb{Z}_+$ such that for $n \geq n_0$,

$B \in \mathcal{S}_n$, $\tau \in E_n$ and $x \in D_{3n}$,

$$\varphi(\tau, x, B) \leq C (\lambda^2(B))^{1-d/4} |\log \lambda^2(B)|^2 \\ = C 2^{-(2-d/2)n} (2n \log 2)^2. \quad (2)$$

We can also take n_0 so that $|X^\tau(s, t)| \leq n_0$ for all $\tau \in [1, \tau_1]$ and $(s, t) \in B_0$ (using the joint continuity of $X^\tau(s, t)$ in τ , s and t). For almost all ω we may argue as follows. For $|x| \geq n_0$ and $\tau \in [1, \tau_1]$, $\varphi(\tau, x, B) = 0$ for all squares $B \subset B_0$. Now take arbitrary fixed $\tau \in [1, \tau_1]$ and x in $\{|x| \leq n_0\}$. For each n greater than n_0 , take elements $\tau(n)$ and $x(n)$ of E_n and D_{3n} respectively such that $|\tau(n) - \tau| < 2^{-3n}$ and $|x(n) - x| < 2^{-3n}$. By the joint Hölder continuity (of any order γ less than $1-d/4$) of $\varphi(\tau, x, B)$, B rectangular (property (iii) in the statement of Theorem 3.2), we have

$$|\varphi(\tau, x, B) - \varphi(\tau(n), x(n), B)| < \text{const. } 2^{-3n\gamma} \quad (n \geq n_0, B \in \mathcal{S}_n) \quad (3)$$

where the constant in (3) depends only on γ , τ_1 and ω . By taking γ so $3\gamma > 2-d/2$, the right hand side of (3) is dominated by that of (2) as $n \rightarrow \infty$. Hence by combining the two we have we have for some n_2 (independent of x and τ) and all $n \geq n_2$,

$$\varphi(\tau, x, B) \leq (C+1) 2^{-(2-d/2)n} (2n \log 2)^2 \\ = (C+1) (\lambda^2(B))^{1-d/4} |\log \lambda^2(B)|^2 \quad (\text{all } n \geq n_2, B \in \mathcal{S}_n). \quad \square$$

4.13 Proof of Corollary 4.9. If B_1 is a compact set in B_0 , then some ϵ -neighbourhood of B_1 is contained in B_0 . For any square B in B_1 of side $h < \epsilon/2$, take ν so $2^{-\nu-1} \leq h < 2^{-\nu}$. Then B is contained in the union of at most four rectangles in \mathcal{S}_ν , and the result then follows easily from Theorem 4.8. \square

4.14 Hölder conditions on the local time of Brownian motion. Let $(B^\tau(\cdot), \tau \geq 0)$ be an Ornstein-Uhlenbeck process in W_0^1 , 1-dimensional Wiener space, with initial distribution given by Wiener measure. Let $\bar{\alpha}(\tau, x, t)$ ($\tau \geq 0$, $x \in \mathbb{R}$, $t \geq 0$) be the jointly continuous function serving as local time of B^τ relative to $[0, t]$, as obtained in Theorem 3.6 (the re-normalisation from the Brownian sheet to the Ornstein-Uhlenbeck process in Wiener space does not affect the Hölder conditions considered here). In section 3.10 we obtained an estimate (****) for the $2k^{\text{th}}$ moments of increments of $\bar{\alpha}$; from this estimate and Corollary 4.4, $\bar{\alpha}$ is Hölder continuous in x or in t of any order less than $1/2$, and in τ of any order less than $1/4$.

In fact, using the more detailed results available about the distribution in path space of $\bar{\alpha}(\tau, \cdot, t)$ and $\alpha(\tau, x, \cdot)$, we may obtain more refined results on the Hölder continuity of α in the x and t directions.

First consider Hölder continuity in the t direction. For fixed τ and x the local time $\bar{\alpha}(\tau, x, \cdot)$ has the same distribution

in $C[0, \infty)$ as the maximal function $b_x^*(t)$ of a Brownian motion $b(t)$ starting at $-|x|$, defined by

$$b_x^*(t) = \max\{0, \max\{b(s) : 0 \leq s \leq t\}\}$$

(see Itô and McKean (1974), page 43). Hence, Lévy's Hölder continuity of Brownian motion on the line gives us a Hölder condition on $\bar{\alpha}(\tau, x, t)$ in t , for fixed τ and x . We now give a quasi-everywhere analogue of this result, where the modulus of continuity is increased only by a constant:

4.15 THEOREM. (a) For fixed x , we have almost surely

$$\lim_{\delta \downarrow 0} \sup_{0 \leq \tau \leq 1} \sup_{\substack{s, t \in [0, 1] \\ |t-s| \leq \delta}} |\bar{\alpha}(\tau, x, t) - \bar{\alpha}(\tau, x, s)| / (|t-s| |\log |t-s||)^{1/2} \leq \sqrt{6}.$$

(b) We have almost surely

$$\lim_{\delta \downarrow 0} \sup_{0 \leq \tau \leq 1} \sup_{\substack{s, t \in [0, 1] \\ |t-s| \leq \delta}} \sup_{x \in \mathbb{R}} |\bar{\alpha}(\tau, x, t) - \bar{\alpha}(\tau, x, s)| / (|t-s| |\log |t-s||)^{1/2} \leq \sqrt{8}.$$

4.16 Remark. Theorem 4.15 implies *a fortiori* that if $\alpha(x, t)$ is the jointly continuous local time of a Brownian motion at x relative to $[0, t]$, then for quasi-every Brownian path

$$\lim_{\delta \downarrow 0} \sup_{\substack{s, t \in [0, 1] \\ |t-s| \leq \delta}} |\alpha(x, t) - \alpha(x, s)| / (|t-s| |\log |t-s||)^{1/2} \leq \sqrt{6}$$

and

$$\lim_{\delta \downarrow 0} \sup_{\substack{s, t \in [0, 1] \\ |t-s| \leq \delta}} \sup_{x \in \mathbb{R}} |\alpha(x, t) - \alpha(x, s)| / (|t-s| |\log |t-s||)^{1/2} \leq \sqrt{8}.$$

The proof of Theorem 4.15 is a routine application of the methods we have already used in this chapter together with the proof of the upper bound on Lévy's modulus of continuity for Brownian motion (McKean (1969), pages 15-16). We give an outline:

4.17 Proof of Theorem 4.15. (a) Let $\lambda > \lambda' > 6$, and $\delta > 0$.

Denote by $A(\tau, x, m)$ the event that

$$|\bar{\alpha}(\tau, x, t) - \bar{\alpha}(\tau, x, s)| > (\lambda (t-s) |\log(t-s)|)^{\frac{1}{2}}$$

for some s and t in $[0, 1]$, $0 \leq (t-s) \leq 2^{(1-\delta)m}$.

The event $A(\tau, x, m)$ has the same probability as the event that the Brownian path $b_x(t)$ satisfies

$$|b_x^*(t) - b_x^*(s)| > (\lambda (t-s) |\log(t-s)|)^{\frac{1}{2}}$$

for some s and t in $[0, 1]$, $0 \leq (t-s) \leq 2^{(1-\delta)m}$. As in McKean (1969), page 16, provided δ is small enough this last event is for large m contained in the event that

$$|b_x(j2^{-n}) - b_x(i2^{-n})| > (\lambda' (j2^{-n} - i2^{-n}) |\log(j2^{-n} - i2^{-n})|)^{\frac{1}{2}}$$

for some $n \geq m$ and some i and j in $\{0, 1, 2, \dots, 2^n\}$ with $0 < (j-i) \leq 2^{\delta n}$.

The probability of this last event is majorised (see McKean (1969), page 15) by

$$\begin{aligned} & \text{const.} \cdot \sum_{n=m}^{\infty} n^{-\frac{1}{2}} 2^{n(1+\delta)} 2^{-n(1-\delta)\lambda'/2} \\ & \leq \text{const.} \cdot 2^{m(1+\delta)} 2^{-m(1-\delta)\lambda'/2}, \end{aligned}$$

so that the probability that $A(\tau, x, m)$ occurs for some τ in $[0, 1] \cap 2^{-(2+\delta)m} \mathbb{Z}$ is at most

$$\text{const.} \cdot 2^{m(1+\delta)} 2^{m(2+\delta)} 2^{-m(1-\delta)\lambda'/2}$$

which is summable in m , provided δ is small enough. Hence by

Borel-Cantelli, there exists m_0 such that for $m \geq m_0$,
 $\tau \in [0,1] \cap 2^{-(4+\delta)} \mathbb{Z}$ and s and t in $[0,1]$ such that
 $0 < t-s \leq 2^{-(1-\delta)m_0}$,

$$|\bar{\alpha}(\tau, x, t) - \bar{\alpha}(\tau, x, s)| \leq (\lambda(t-s)|\log(t-s)|)^{\frac{1}{2}}.$$

Using the Hölder continuity of $\bar{\alpha}$ of any order less than $1/4$ in the τ direction, we can now fill in the other values of τ much as in earlier proofs in this chapter.

(b) This is similar to the proof of (a), but we now need to obtain a Hölder condition in t holding uniformly in τ and x . We can use the same methods, now filling in all values of (τ, x) from those on a sequence of lattices in the (τ, x) -plane of fineness $2^{-(2+\delta)n}$ in τ and $2^{-(1+\delta)}$ in x . This means we now require λ to be greater than 8. We are able to get the result uniformly over all x (rather than just over x in a bounded interval) because the local time is zero outside image set $\{B^\tau(t): 0 \leq t \leq 1, 0 \leq \tau \leq 1\}$, which is bounded. \square

As for the Hölder continuity in the x direction of the local time, a similar adaptation of the proof of the "almost everywhere" result of McKean (1962) (which uses Tanaka's formula for the local time to estimate the probability that increments are large) gives us the following:

4.18 THEOREM. If $\alpha(x, t)$ denotes the jointly continuous local time of a Brownian path in \mathbb{R} at x relative to $[0, t]$, then for quasi-every path

$$\lim_{\delta \downarrow 0} \sup_{\substack{x, y \in \mathbb{R} \\ |y-x| \leq \delta}} |\alpha(x, t) - \alpha(y, t)| / (\delta |\log \delta|)^{\frac{1}{2}} = 2(6 \sup_{z \in \mathbb{R}} \alpha(z, t))^{\frac{1}{2}}.$$

CHAPTER 5 THE PROBABILITY DISTRIBUTION OF INTERSECTION LOCAL TIME

In this chapter we prove some results on the probability distribution of the intersection local time of two independent Brownian motions starting at the origin, introduced by Geman, Horowitz and Rosen (1984). In particular we consider the probability that the local time (relative to the unit square in the time domain) at zero is very small. A scaling argument allows one to relate these results to results on the local time relative to small (or large) squares in the time domain.

5.1 Preliminaries. We are here concerned with probability distributions; for two random variables X_1 and X_2 , we shall say that $X_1 \overset{L}{\geq} X_2$ (or $X_2 \overset{L}{\leq} X_1$) if for all $x \in \mathbb{R}$, $P(X_1 > x) \geq P(X_2 > x)$. We shall say that $X_1 \overset{L}{=} X_2$ if $X_1 \overset{L}{\geq} X_2$ and $X_2 \overset{L}{\geq} X_1$.

Let d be 2 or 3, and let b_1 and b_2 be independent Brownian motions in \mathbb{R}^d . For Borel $B \subset \mathbb{R}_+^2$, denote by

$$\alpha(x, B), \quad x \in \mathbb{R}^d$$

the continuous version of the intersection local time relative to B of $b_1(\cdot)$ and $b_2(\cdot)$, as defined in section 2.2. Such a local time exists almost surely (see Remark 2.26).

For $h > 0$, define $Q_h = (0, h) \times (0, h) \subset \mathbb{R}_+^2$.

We have the following scaling property:

5.2 LEMMA. for all $\tau \geq 1$ and $h > 0$, the random function

$$h^2 (\tau h)^{-d/2} \alpha((\tau h)^{-1/2} x, Q_1) \quad x \in \mathbb{R}^d$$

has the same distribution in $C(\mathbb{R}^d)$ as the continuous version of the intersection local time, relative to Q_h , of the paths

$$b_1^\tau(\cdot) \quad \text{and} \quad b_2^\tau(\cdot)$$

where $(b_1^\tau(t), \tau \geq 0, t \geq 0)$ and $(b_2^\tau(t), \tau \geq 0, t \geq 0)$ are independent 2-parameter Wiener processes in \mathbb{R}^d .

Proof. By the scaling property of the Brownian sheet (section 1.5), the random functions

$$b_2^\tau(t) - b_1^\tau(s) \quad (s \geq 0, t \geq 0)$$

and $(\tau h)^{1/2} (b_2(t) - b_1(s)) \quad (s \geq 0, t \geq 0)$

have the same distribution, considered as random elements of $C(\mathbb{R}_+^2)$.

The result follows by a change of variables. \square

The following is a special case of Lemma 5.2:

5.3 LEMMA. For all $x \in \mathbb{R}^d$ and $h > 0$,

$$h^{2-d/2} \alpha(h^{-1/2} x, Q_1) \stackrel{L}{=} \alpha(x, Q_h).$$

5.4 LEMMA. For $h > 0$ and $x \in \mathbb{R}^d$,

$$E [\alpha(x, Q_h)] = (2\pi)^{-d} \int_{\mathbb{R}^d} \cos(u \cdot x) (2 \{1 - \exp(-h|u|^2/2)\} / |u|^2)^2 du.$$

(1)

In particular, $P[\alpha(x, Q_h) > 0] > 0$.

Proof. α can be expressed by the formal Fourier inversion formula

$$\alpha(x, Q_h) = (2\pi)^{-d} \int_{u \in \mathbb{R}^d} \int_{(s,t) \in Q_h} \exp\{iu \cdot (b_2(t) - b_1(s) - x)\} ds dt du$$

in the sense of Lemma 2.10. Taking expectations, we have

$$E[\alpha(x, Q_h)] = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-iu \cdot x} \left(\int_0^h \int_0^h \exp(-|u|^2(t+s)/2) ds dt \right) du$$

where taking the expectation inside the improper integral is justified because the improper integral is shorthand for the L^{2k} limit of a sequence of integrals over bounded regions. Hence, since α is real, (1) follows. The integrand in the right hand side of (1) is the product of $\cos(u \cdot x)$ and a strictly decreasing function of $|u|$. It follows that $E[\alpha(x, Q_h)] > 0$, so that $P[\alpha(x, Q_h) > 0] > 0$. \square

5.5 Remark. The right hand side of the expression (1) in Lemma 5.4 can actually be evaluated by using spherical (when $d = 3$) polar co-ordinates to reduce to an integral over $(0, \infty)$. The resulting expression is not very illuminating, however.

5.6 LEMMA. (Geman, Howowitz and Rosen (1984)) For all $h > 0$, $P[\alpha(0, Q_h) = 0] = 0$.

Proof. By the scaling property (Lemma 5.3), the probability in question is independent of h . But

$$P[\alpha(0, Q_{2h})=0] \leq P[\alpha(0, Q_h)=0, \alpha(0, (h, 2h) \times (h, 2h))=0].$$

Now condition on $\Sigma\{b_1(t), b_2(s): 0 \leq s, t \leq h\}$. By the Markov property of Brownian motion, $\alpha(0, (h, 2h) \times (h, 2h))$ depends on this sigma-algebra only via $(b_2(h) - b_1(h))$ (see Lemma 5.5 of Geman, Horowitz and Rosen (1984)). Applying Lemma 5.4 gives us

$$P[\alpha(0, Q_{2h})=0] < P[\alpha(0, Q_h)=0] \text{ unless } P[\alpha(0, Q_h)=0] = 0. \quad \square$$

Our next result, which we prove by a coupling argument, states that $\alpha(\cdot, Q_h)$ is stochastically monotone in h :

5.7 PROPOSITION. Let $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$. if $|x| \geq |y|$, then

$$\alpha(y, Q_h) \stackrel{L}{\geq} \alpha(x, Q_h) \text{ for all } h > 0.$$

Proof. By the rotational invariance of Brownian motion, if $|x| = |y|$

then $\alpha(y, Q_h) \stackrel{L}{=} \alpha(x, Q_h)$. Hence it suffices to consider the case $|x| > 0$, $y = \lambda x$ for $0 \leq \lambda < 1$.

Given two paths b and $b': \mathbb{R}_+ \rightarrow \mathbb{R}^d$ with a continuous intersection local time, relative to a set B in \mathbb{R}_+^2 , denote by $\beta(b, b', B)$ the value of this intersection local time at zero.

We are required to prove that if $b(\cdot)$, $b_x(\cdot)$ and $b_y(\cdot)$ are Brownian motions starting at 0, x and y respectively (where $y = \lambda x$ and $0 \leq \lambda < 1$), and if the path b is independent of the paths b_x and b_y , then

$$\beta(b, b_x, Q_h) \stackrel{L}{\leq} \beta(b, b_y, Q_h).$$

Let $P = \{z \in \mathbb{R}^d : |z-x| = |z-y|\}$. Let $\rho: \mathbb{R}^d \rightarrow \mathbb{R}^d$ be reflection in P .

Without loss of generality, b_y is coupled to b_x by defining the stopping time $T = \inf\{t: b_x(t) \in P\}$, and setting

$$\begin{aligned} b_y(t) &= \rho(b_x(t)) & t \leq T \\ b_y(t) &= b_x(t) & t \geq T. \end{aligned}$$

Then b_x and b_y are identical after time T , so it suffices to prove that, setting $T_h = \min\{T, h\}$,

$$\beta(b, b_x, [0, h] \times [0, T_h]) \stackrel{L}{\leq} \beta(b, b_y, [0, h] \times [0, T_h]).$$

Define the stopping times $S = \inf\{s: b(s) \in P\}$, $S_h = \min\{S, h\}$, and the path \bar{b} by

$$\begin{aligned} \bar{b}(s) &= b(s) & s \leq S \\ \bar{b}(s) &= \rho(b(s)) & s \geq S. \end{aligned}$$

Then the random path $\bar{b}(\cdot)$ has the same law in $C([0, \infty) \rightarrow \mathbb{R}^d)$ as $b(\cdot)$. By definition of S and T , the origin is almost surely not in the compact set $\{b(s) - b_x(t) : 0 \leq s \leq S, 0 \leq t \leq T\}$, so that we have almost surely

$$\beta(b, b_x, [0, S_h] \times [0, T_h]) = 0.$$

Hence we have almost surely

$$\begin{aligned}\beta(b, b_x, [0, h] \times [0, T_h]) &= \beta(b, b_x, [S_h, h] \times [0, T_h]) \\ &= \beta(\bar{b}, b_y, [S_h, h] \times [0, T_h])\end{aligned}$$

by the definitions of \bar{b} and b_y as reflections. Hence (since \bar{b} and b have the same law in $C([0, \infty) \rightarrow \mathbb{R}^d)$),

$$\begin{aligned}\beta(b, b_x, [0, h] \times [0, T_h]) &\stackrel{L}{=} \beta(b, b_y, [S_h, h] \times [0, T_h]) \\ &\leq \beta(b, b_y, [0, h] \times [0, T_h])\end{aligned}$$

as desired. □

We are now able to obtain an estimate (possibly not sharp) for the rate at which the probability that $\alpha(0, Q_1) < \epsilon_n$ tends to zero, for a particular sequence ϵ_n tending to zero. This will be needed in the next chapter, when we prove that the self-intersection local time of quasi-every path is strictly positive.

5.8 LEMMA. There exist finite positive constants c_1 , c_2 and c_3 such that if we set

$$\epsilon_n = c_1^n (n!)^{-2+d/2}$$

then

$$P[\alpha(0, Q_1) < \epsilon_n] < c_2 \exp(-c_3 n). \quad (*)$$

5.9 Remark. The right hand side of (*) is majorised for all n by

$$c_4(\epsilon_n)^{c_5/\log|\log \epsilon_n|},$$

for some positive finite constants c_4 and c_5 . It is an open question whether there are constants c_4 and c_5 such that

$$P[\alpha(0, Q_1) > \epsilon] = O\left[c_4 \epsilon^{c_5/\log|\log \epsilon|} \right] \quad (\epsilon \rightarrow 0).$$

5.10 Proof of Lemma 5.8. Let $a_n = \lambda^n n!$, where λ is a finite positive constant to be chosen later. Let

$$P_n = P[\alpha(0, Q_{a_n}) < 1].$$

By the scaling property (Lemma 5.3),

$$P_n = P[\alpha(0, Q_1) < a_n^{-2+d/2}]. \quad (1)$$

Define the square subsets A_n , R_n and D_n of \mathbb{R}_+^2 by

$$A_n = Q_{a_n} = (0, a_n) \times (0, a_n)$$

$$R_n = (a_{n-1}, a_n) \times (a_{n-1}, a_n)$$

$$D_n = Q_{(a_n - a_{n-1})}.$$

Let Σ_n be the σ -algebra generated by $\{(b_1(s), b_2(t)) : (s, t) \in A_n\}$.

Let

$$U_n = b_2(a_n) - b_1(a_n).$$

Then for all $n \geq 1$, the event $\{\alpha(0, A_{n+1}) < 1\}$ is contained in the union of the events

$$\{|U_n| > a_{n+1}^{1/2}\} \quad \text{and} \\ \{|U_n| \leq a_{n+1}^{1/2}\} \cap \{\alpha(0, A_n) < 1\} \cap \{\alpha(0, R_{n+1}) < 1\}.$$

Taking probabilities and conditioning on Σ_n , we have

$$P_{n+1} \leq P[|U_n| > a_{n+1}^{1/2}] + \int_{\Omega_n} P[\alpha(0, R_{n+1}) < 1 \mid \Sigma_n] dP, \quad (2)$$

where $\Omega_n = \{|U_n| \leq a_{n+1}^{1/2}\} \cap \{\alpha(0, A_n) < 1\} \in \Sigma_n$.

$\alpha(0, R_{n+1})$ depends on Σ_n only via U_n . So for almost all $\omega \in \Omega_n$,

$$P[\{|\alpha(0, R_{n+1})| < 1\} \mid \Sigma_n] = P[|\alpha(x, D_{n+1})| < 1]$$

where $x = -U_n$, so $|x| \leq a_{n+1}^{1/2}$ for $\omega \in \Omega_n$. But for all $x \in \mathbb{R}^d$, the scaling property (Lemma 5.3) implies that

$$P[\alpha(x, D_{n+1}) < 1] = P[\alpha((a_{n+1} - a_n)^{-1/2} x, Q_1) < (a_{n+1} - a_n)^{-2+d/2}]. \quad (3)$$

Since $\alpha(x, Q_1)$ is stochastically decreasing in $|x|$ (Proposition 5.7), for large n and all x such that $|x| \leq a_{n+1}^{1/2}$, the right hand side of (3) is at most

$$P[\alpha(x_0, Q_1) < (a_{n+1} - a_n)^{-2+d/2}]$$

where x_0 is an arbitrary fixed vector in \mathbb{R}^d of length greater than 1, so that for large n , $|x_0| > [a_{n+1}/(a_{n+1} - a_n)]^{1/2}$.

Now $(a_{n+1} - a_n)^{-2+d/2} \rightarrow 0$ as $n \rightarrow \infty$, so since $\alpha(x_0, Q_1) > 0$ with positive probability (Lemma 5.4), there exists $c > 0$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} P[\alpha(x_0, Q_1) < (a_{n+1} - a_n)^{-2+d/2}] &= P[\alpha(x_0, Q_1) = 0] \\ &< e^{-c}. \end{aligned}$$

Hence for all large enough n and almost all $\omega \in \Omega_n$, the integrand in the second term of the right hand side of (2) is less than e^{-c} , so for large enough n ,

$$\begin{aligned} \int_{\Omega_n} P[\alpha(0, R_{n+1}) < 1 | \Sigma_n] &< e^{-c} \cdot P(\Omega_n) \\ &\leq e^{-c} \cdot P_n. \end{aligned} \quad (4)$$

As for the first term in the right hand side of (2),

$$P[|b_2(a_n) - b_1(a_n)| > a_{n+1}^{1/2}] \leq \text{const.} \int_{u_n}^{\infty} x^2 \exp(-x^2/2) dx$$

where $u_n = (a_{n+1}/2a_n)^{1/2} = (\lambda/2)^{1/2} (n+1)^{1/2}$. So integrating by parts,

$$P[|b_2(a_n) - b_1(a_n)| > a_{n+1}^{1/2}] \leq \text{const.} (n+1)^{1/2} \exp\{-\lambda(n+1)/4\} \quad (n \rightarrow \infty). \quad (5)$$

Applying the estimates (4) and (5) to (2), we obtain

$$P_{n+1} \leq c' (n+1)^{1/2} \exp\{-\lambda(n+1)/4\} + e^{-c} P_n \quad (n \text{ large}). \quad (6)$$

Let $Q_n = e^{nc} \cdot P_n$. Applying (6), we find that $\sum_1^{\infty} (Q_{n+1} - Q_n)_+ < \infty$, provided that $\lambda > 4c$. Hence $\{Q_n : n \geq 1\}$ is bounded and there exists $c' > 0$ such that $P_n \leq c' e^{-cn}$. By (1), the result (*) is proved with $\epsilon_n = (\lambda^{-2+d/2})^n (n!)^{-2+d/2}$.

We now resume our discussion of quasi-everywhere properties of Brownian local time and self-intersection local time, as a means of describing the quasi-everywhere properties of Brownian level sets. According to Lemma 2.5, the local time can provide a lower bound for a Hausdorff measure of a level set. This is useful only if the local time is strictly positive; hence we are here concerned with showing that the (self-intersection) local time is strictly positive for quasi-every path.

In contrast with earlier chapters, the problems posed by the cases of self-intersection local time and ordinary local time are distinct. In the case of self-intersection local time the main problem is the fact that the probability distribution of the local time is not well understood; hence the need for the work in the last chapter. In the case of ordinary local time the distribution of the local time is fully understood, but it is harder to construct independent events when there is only one time-parameter.

First consider self-intersection local time. We shall find an open bounded B_0 in the upper triangle such that the self-intersection local time of a Brownian path relative to B_0 exists and is strictly positive for quasi-every path.

6.1 REMARK. It is easy to see from Theorem 3.2 that for quasi-every path in \mathbb{R}^d ($d=2$ or $d=3$), the self-intersection local time relative to the unbounded upper triangle is strictly positive. Set

$$A_n = (n, n+\frac{1}{2}) \times (n+\frac{1}{2}, n+1) \quad (n = 1, 2, 3, \dots).$$

Let $(B_\tau(\cdot))_{\tau \geq 0}$ denote an Ornstein-Uhlenbeck process in W_0^d , with initial distribution given by Wiener measure. Denote by $\alpha(\tau, \cdot, A_n)$ the (jointly continuous) self-intersection local time of $B_\tau(\cdot)$ relative to A_n , as obtained in Theorem 3.2. Then $\alpha(\tau, 0, A_1)$ is continuous in τ and $\alpha(0, 0, A_1) > 0$ with positive probability (see Lemma 5.4). So for some $h > 0$,

$$P\{\alpha(\tau, 0, A_1) > 0, \tau \in [0, h]\} > 0.$$

For $n = 2, 3, 4, \dots$, the processes $(\alpha(\tau, 0, A_n), \tau \geq 0)$ are independent copies of $(\alpha(\tau, 0, A_1), \tau \geq 0)$. Hence

$$P\left(\bigcup_{n=1}^{\infty} \{\alpha(\tau, 0, A_n) > 0, \tau \in [0, h]\}\right) = 1,$$

and similarly for the intervals $[h, 2h]$, $[2h, 3h]$, and so on. Hence for quasi-every Brownian path there exists n such that the self-intersection local time $\alpha(x, A_n)$ satisfies

$$\alpha(0, A_n) > 0.$$

Thus, by the arguments of Chapter 7 below, we can obtain

$$\dim\{x: x = b(t) = b(s), \text{ some distinct } s, t \text{ in } [0, \infty)\} = 4-d \quad (1)$$

for quasi-every path $b(\cdot)$ in \mathbb{R}^d , without recourse to the harder results of Chapter 5. However, if in (1) we wish to restrict s and t from $[0, \infty)$ to $[0, 1]$, then the familiar scaling arguments of the "almost everywhere" theory do not carry over to the "quasi-everywhere" theory. Hence we need a different argument for bounded time-sets.

6.2 THEOREM. Let $d=2$ or $d=3$. There exists an open set B_0 in the triangular set $\Delta = \{(s,t): 0 \leq s < t < 1\}$, such that B_0 satisfies the hypothesis of Theorem 3.6, and setting self-intersection local times ϕ as in that theorem and φ as in Theorem 3.2:

$$\varphi(\tau, x, B_0) = \phi(\tau, x) \quad (\text{all } x \in \mathbb{R}^d, \tau \geq 1), \quad \text{almost surely}$$

and $\varphi(\tau, 0, B_0) = \phi(\tau, 0) > 0 \quad (\tau \geq 1), \quad \text{almost surely.}$

Proof. B_0 is defined as follows. Let

$$B_n = (2 \times 2^{-n}, 3 \times 2^{-n}) \times (3 \times 2^{-n}, 4 \times 2^{-n}) \quad (n=2, 3, 4, \dots).$$

and
$$B_0 = \bigcup_{n=2}^{\infty} B_n.$$

Suppose $(b^\tau(t), t \geq 0, \tau \geq 0)$ is a Brownian sheet in \mathbb{R}^d and (as in earlier sections) we set

$$X^\tau(s, t) = b^\tau(t) - b^\tau(s), \quad X(s, t) = X^1(s, t).$$

By Lemmas 2.16 and 2.23, for $k \in \mathbb{Z}_+$ and $0 \leq \gamma < 2-d/2$, the estimating integral $J_X(2k, \gamma, B_0)$ satisfies

$$(J_X(2k, \gamma, B_0))^{1/2k} \leq \sum_{n=2}^{\infty} (J_X(2k, \gamma, B_n))^{1/2k}$$

$$\leq c \sum_{n=2}^{\infty} (\lambda^2(B_n))^{(2-(d/2)-\gamma)/2}$$

$$< \infty$$

so B_0 satisfies the hypothesis of Theorem 3.6. Let

$$(\varphi(\tau, \cdot, B), \tau \geq 1) \quad (B \text{ Borel in } \Delta)$$

and $(\phi(\cdot, \tau), \tau \geq 1)$

be the self-intersection local times of X^τ obtained by applying Theorems 3.2 and 3.6 respectively (using this particular B_0).

We now prove that given $\tau_1 \geq 1$, $\varphi(\tau, x, B_0)$ is finite and continuous in x for all $\tau \in [1, \tau_1]$, almost surely; for almost

all fixed ω the following holds for all $\tau \in [1, \tau_1]$.

Since $\varphi(\tau, x, \cdot)$ is a measure (property (ii) in Theorem 3.2), we have for all $x \in \mathbb{R}^d$

$$\varphi(\tau, x, B_0) = \sum_2^\infty \varphi(\tau, x, B_n).$$

By our uniform Hölder estimate of $\varphi(\tau, x, B)$ for dyadic rectangles B (Theorem 4.8), and the definition of B_n ,

$$\sum_2^\infty \sup\{\varphi(\tau, x, B_n) : x \in \mathbb{R}^d\} \leq \infty.$$

Moreover, for all n $\varphi(\tau, \cdot, B_n)$ is continuous by Theorem 3.2 (property (iii)), because B_n is a rectangle. Hence, $\varphi(\tau, \cdot, B_0)$ is continuous since it is a uniform limit of continuous functions.

Thus $\phi(\tau, \cdot)$ and $\varphi(\tau, \cdot, B_0)$ are both continuous versions of the local time relative to B_0 of X^τ ; hence they are identical, i.e. $\varphi(\tau, x, B_0) = \phi(\tau, x)$ for all x in \mathbb{R}^d , $\tau \in [1, \tau_1]$. In particular, $\varphi(\cdot, 0, B_0)$ is Hölder continuous of any order less than $1-d/4$.

We now prove that $\varphi(\tau, 0, B_0) > 0$ for all $\tau \geq 1$, almost surely. We do this by using the Hölder continuity of $\varphi(\tau, 0, B_0)$ in τ ; the argument is related to those we used in Chapter 4.

It suffices to prove that for $\tau_1 > 1$,

$$\varphi(\tau, 0, B_0) > 0 \quad (1 \leq \tau \leq \tau_1), \quad \text{a.s.}$$

$\varphi(\tau, 0, B_n)$ can be viewed as the intersection local time at the origin of two independent Brownian motions scaled by $\tau^{1/2}$, relative to the square $(0, 2^{-n}) \times (0, 2^{-n})$. Using the notation of Chapter 5, and the

scaling property (Lemma 5.2),

$$\begin{aligned}\varphi(\tau, 0, B_n) &\stackrel{L}{=} \tau^{-d/2} 2^{-n(2-d/2)} \alpha(0, Q_1) \\ &\stackrel{L}{\geq} \tau_1^{-d/2} 2^{-n(2-d/2)} \alpha(0, Q_1)\end{aligned}\quad (1)$$

where $\alpha(0, Q_1)$ is the intersection local time of two Brownian motions relative to the unit square $(0,1) \times (0,1)$.

Let ϵ_n be as defined in Lemma 5.8, i.e. $\epsilon_n = c_1^n (n!)^{-2+d/2}$, some suitable $c_1 > 0$. Let $\delta_n = 2^{-n(2-d/2)} \tau_1^{-d/2} \epsilon_n$. Hence

$$\begin{aligned}P[\varphi(\tau, 0, B_n) < \delta_n] &\leq P[\alpha(0, Q_1) < \epsilon_n] \\ &< c_2 \exp(-c_3 n) \quad (\text{all } n \geq 2, \quad 1 \leq \tau \leq \tau_1)\end{aligned}$$

(where c_2 and c_3 are positive constants) by Lemma 5.8.

The random variables $\varphi(\tau, 0, B_n)$ ($n \geq 2$) are mutually independent and for $j \leq n$, $\varphi(\tau, 0, B_j) \stackrel{L}{\geq} \varphi(\tau, 0, B_n)$. So for all $\tau \in [1, \tau_1]$ and $n \geq 2$,

$$\begin{aligned}P[\varphi(\tau, 0, B_0) < \delta_n] &\leq P[\varphi(\tau, 0, B_j) < \delta_n, \quad 2 \leq j \leq n] \\ &= \prod_{j=2}^n P[\varphi(\tau, 0, B_j) < \delta_n] \\ &\leq (P[\varphi(\tau, 0, B_n) < \delta_n])^{n-1} \\ &\leq c_2^{n-1} \exp(-c_3(n^2-n))\end{aligned}\quad (2)$$

where the constants c_2 and c_3 are as in Lemma 5.8. Let F_n be the set of all τ in $[1, \tau_1]$ of the form $\tau = m\delta_n^5$, where $m \in \mathbb{Z}$.

Then by (2)

$$\begin{aligned}P \bigcup_{\tau \in F_n} \{\varphi(\tau, 0, B_0) < \delta_n\} &\leq (\tau_1 - 1) \delta_n^{-5} \sup_{1 \leq \tau \leq \tau_1} P[\varphi(\tau, 0, B_0) < \delta_n] \\ &\leq \text{const.} \cdot (\text{const.})^n \cdot (n!)^{5(2-d/2)} \cdot \exp(-c_3(n^2-n)).\end{aligned}\quad (3)$$

The right hand side of (3) is summable in n . By Borel-Cantelli, there exists a.s. some n_0 such that

$$\varphi(\tau, 0, B_0) \geq \delta_n \quad (\text{all } n \geq n_0, \tau \in F_n). \quad (4)$$

But $\varphi(\cdot, 0, B_0)$ is almost surely Hölder continuous of any order less than $1-d/4$. Hence we have almost surely:

$$\lim_{\delta \rightarrow 0} \sup \{ |\varphi(\tau, 0, B_0) - \varphi(\sigma, 0, B_0)| / |\tau - \sigma|^5 : 1 \leq \tau \leq \sigma \leq \tau_1, |\tau - \sigma| < \delta \} = 0. \quad (5)$$

Together, (4) and (5) imply that for all $\tau \in [1, \tau_1]$, $\varphi(\tau, 0, B_0) > 0$, so the theorem is proved. \square

We now give the corresponding result for ordinary Brownian local time, that this is strictly positive for quasi-every path:

6.3 THEOREM. Let $(b^\tau(t), t \geq 0, \tau \geq 0)$ be a Brownian sheet taking values in \mathbb{R} . Let $\alpha(\tau, x, t)$ be the (continuous) local time at x of the path $b^\tau(\cdot)$ relative to $[0, t]$, as in Theorem 3.8. Then with probability 1,

$$\alpha(\tau, 0, 1) > 0 \quad (\tau > 0).$$

To prove this we need the following result by Walsh (1982):

6.4 LEMMA. With probability 1,

$$\limsup_{t \downarrow 0} b^\tau(t)/(t \log |\log t|)^{\frac{1}{2}} = \tau^{\frac{1}{2}} \quad (\tau > 0).$$

$$\liminf_{t \downarrow 0} b^\tau(t)/(t \log |\log t|)^{\frac{1}{2}} = -\tau^{\frac{1}{2}} \quad (\tau > 0).$$

In particular, 0 is an accumulation point of $(b^\tau)^{-1}(0)$ for all $\tau > 0$.

6.5 Remark. Fukushima (1984) gives an alternative, potential-theoretic proof for the equivalent result that quasi-every Brownian motion satisfies the law of the iterated logarithm.

6.6 Proof of Theorem 6.3. Let $0 < \tau_1 < \tau_2 < \infty$. It suffices to prove that

$$\alpha(\tau, 0, 1) > 0 \quad \tau \in [\tau_1, \tau_2], \quad \text{a.s.}$$

Given $0 < \varepsilon < \varepsilon'/2$, define the event $A(\varepsilon, \varepsilon')$ by

$$A(\varepsilon, \varepsilon') = \{ \exists \tau \in [\tau_1, \tau_2] \text{ such that } b^\tau(t) \neq 0, \quad t \in [\varepsilon, \varepsilon'/2] \}.$$

$A(\varepsilon, \varepsilon')$ is P -measurable (easily verified). Given $\varepsilon' > 0$, the events $A(\varepsilon, \varepsilon')$ decrease as ε decreases. By Lemma 6.4 and the continuity of measure,

$$\lim_{\varepsilon \downarrow 0} P(A_{\varepsilon, \varepsilon'}) = 0. \quad (1)$$

Let $\delta > 0$. By (1) we can find $t_1, \dots, t_7 \in (0, \infty)$ such that $t_7 = 1$ and for $i = 6, 5, 4, 3, 2, 1$

$$t_i < t_{i+1}/2$$

and

$$P(A(t_i, t_{i+1})) < \delta$$

Hence, defining the event $\Omega_0 = \bigcup_{i=1}^6 A(t_i, t_{i+1})$, we have

$P(\Omega_0) < 6\delta$. Since δ is arbitrary it suffices to show that for almost all $\omega \notin \Omega_0$, the local time $\alpha(\tau, 0, 1)$ is strictly positive for all $\tau \in [\tau_1, \tau_2]$. We do this using the Hölder continuity in τ of the local time. For $\tau > 0$, $\varepsilon > 0$ define the event $E(\tau, \varepsilon)$ by

$$E(\tau, \varepsilon) = \Omega_0 \cap \{\alpha(\tau, 0, 1) < \varepsilon\}$$

For $1 \leq i \leq 7$, and $\tau_1 \leq \tau \leq \tau_2$, set $T_i^\tau = \inf\{t \geq t_i : b^\tau(t) = 0\}$, so that T_i^τ is a stopping time relative to the natural filtration of $b^\tau(\cdot)$, and by definition $\Omega_0 \subset \bigcap_{i=1}^6 \{T_i^\tau \leq t_{i+1}/2\}$.

Define events $E_i(\tau)$ by

$$E_i(\tau, \varepsilon) = \{\alpha(\tau, 0, T_i^\tau + t_{i+1}/2) - \alpha(\tau, 0, T_i^\tau) < \varepsilon\} \cap \{T_{i+1}^\tau \leq t_{i+1}/2\}.$$

Then $E(\tau, \varepsilon) \subset \bigcap_{i=1}^6 E_i(\tau, \varepsilon)$. Moreover, by the strong Markov property of Brownian motion, E_{i+1} is independent of E_i ($1 \leq i \leq 6$), so

$$P[E(\tau, \varepsilon)] \leq \prod_{i=1}^6 P[E_i(\tau, \varepsilon)]. \quad (2)$$

Again by the strong Markov property, the distribution of $[\alpha(\tau, 0, T_i^\tau + \cdot) - \alpha(\tau, 0, T_i^\tau)]$ is the same as that of $\alpha(\tau, 0, \cdot)$, i.e. the local time at zero of $\tau^{1/2}$ times a standard Brownian local time at zero. By a simple change of variables this is the same distribution as that of $\tau^{-1/2} \alpha(1, 0, \cdot)$. This in turn is the same distribution (see Itô and McKean (1974), page 43) as that of $b^*(\cdot)$ where $b(t)$ is standard Brownian motion on the line and

$$b^*(t) = \max_{0 \leq s \leq t} b(s).$$

Hence

$$P[E_i(\tau, \varepsilon)] \leq P[\tau^{-1/2} b^*(t_{i+1}/2) < \varepsilon].$$

By the reflection principle,

$$\begin{aligned} P[E_i(\tau, \epsilon)] &\leq 1 - 2P[b(t_{i+1}/2) > \tau^{\frac{1}{2}} \epsilon] \\ &\leq 2(\pi t_{i+1})^{-\frac{1}{2}} \tau^{\frac{1}{2}} \epsilon. \end{aligned}$$

We have fixed t_i ($1 \leq i \leq 7$) and τ_2 . Hence (3) gives us

$$P[E(\tau, \epsilon)] \leq \text{const.} \times \epsilon^6.$$

So

$$P\{E(n\epsilon^5, \epsilon), \text{ some integer } n \in [\tau_1\epsilon^{-5}, \tau_2\epsilon^{-5}]\} \leq \text{const.} \cdot \epsilon^6 \epsilon^{-5}$$

Choose a sequence $\epsilon_k \downarrow 0$ such that $\sum_k \epsilon_k$ is a convergent series.

By Borel-Cantelli and the definition of E , for almost all ω outside Ω_0 , there exists k_0 such that for all $k \geq k_0$,

$$\alpha(n\epsilon_k^5, 0, 1) \geq \epsilon_k, \quad \text{all } n \in [\tau_1\epsilon_k^{-5}, \tau_2\epsilon_k^{-5}] \cap \mathbb{Z}. \quad (3)$$

But $\alpha(\tau, 0, 1)$ is Hölder continuous in τ of any order less than $1 - d/4$ (Theorem 3.3). Given $\tau \in [\tau_1, \tau_2]$ we may take $n(k) \in [\tau_1\epsilon_k^{-5}, \tau_2\epsilon_k^{-5}]$, such that $|n(k)\epsilon_k^5 - \tau| \leq \epsilon(k)^5$. The Hölder continuity in τ of α then gives us

$$|\alpha(\tau, 0, 1) - \alpha(n(k)\epsilon_k^5)| = o(\epsilon_k) \quad \text{as } k \rightarrow \infty. \quad (4)$$

comparing (3) and (4) shows that $\alpha(\tau, 0, 1) > 0$. This argument holds for all $\tau \in [\tau_1, \tau_2]$, for almost all $\omega \notin \Omega_0$, as required. \square

In the last chapter we showed that the self-intersection local time of a Brownian path in \mathbb{R}^3 (respectively the local time at zero of a Brownian path in \mathbb{R}) is strictly positive for quasi-every path. Using this and the Hölder conditions on the local time obtained for quasi-every path in chapter 4, we now deduce quasi-everywhere results about the Hausdorff dimensions of level sets. That is, we obtain quasi-everywhere results on the dimension of the set of pairs of distinct times whose Brownian images in \mathbb{R}^3 coincide (respectively the inverse image of $\{0\}$ under a Brownian path in \mathbb{R}). Set $\Delta = \{(s,t): 0 \leq s < t \leq 1\}$.

The following theorem implies *a fortiori* that quasi-every Brownian path in \mathbb{R}^3 (or \mathbb{R}^2) intersects itself in the unit time-interval (or in any time-interval), thus strengthening the well-known result of Dvoretzky, Erdős and Kakutani (1950) to a quasi-everywhere result.

7.1 THEOREM. The following holds for quasi-every path $b(\cdot)$ in W_0^d ($d=2$ or $d=3$):

$$\dim\{(s,t) \in \Delta: b(s) = b(t)\} = 2-d/2.$$

Proof. It suffices to prove that for any $\tau_1 \geq 1$, we have for almost every Brownian sheet $(b^\tau(t), t \geq 0, \tau \geq 0)$ in \mathbb{R}^d ,

$$\dim((X^\tau)^{-1}(0) \cap \Delta) = 2-d/2 \quad (1 \leq \tau \leq \tau_1). \quad (1)$$

Let B_0 be the set described in Theorem 6.2, and set φ as in Theorem 3.2. We can argue as follows for almost all fixed ω and

all fixed $\tau \in [1, \tau_1]$. By Theorem 6.2, $\varphi(\tau, 0, B_0) > 0$. Hence there exists open B_1 with closure contained in B_0 , such that $\varphi(\tau, 0, B_1) > 0$.

By our Hölder condition on φ in the set variable B (Corollary 4.9), for any set $B \subset B_1$ of sufficiently small diameter $e(B)$,

$$\varphi(\tau, 0, B) \leq \text{const.} \cdot h_1(e(B))$$

where
$$h_1(u) = u^{2-d/2} |\log u|^2.$$

Hence by the (nonstochastic) real function theory of Chapter 2 (Lemma 2.5)

$$h_1 - m(B_1 \cap (X^\tau)^{-1}(0)) > 0 \quad (*)$$

so
$$\dim [(X^\tau)^{-1}(0) \cap \Delta] \geq 2-d/2.$$

On the other hand, suppose B is a rectangle in the upper triangle Δ (so for all τ , X^τ has a continuous self-intersection local time relative to B by Theorem 3.2). By the Hölder continuity of the Brownian sheet (Theorem 1.6), we have for all $\tau \in [1, \tau_1]$

$$|X^\tau(s) - X^\tau(t)| \leq \text{const.} \cdot g(|s-t|) \quad 0 \leq s \leq t \leq 1$$

where $g(u) = (u |\log u|)^{1/2}$. Hence by Lemma 2.6 we have

$$h_2 - m[B \cap (X^\tau)^{-1}(0)] < \infty \quad (**)$$

where
$$h_2(u) = u^2 (g(u))^{-d} = u^{2-d/2} |\log u|^{-d/2},$$

and hence
$$\dim [B \cap (X^\tau)^{-1}(0)] \leq 2-d/2.$$

Since Δ is a countable union of such B , (1) follows. \square

The next result is an analogue to Theorem 7.1 for Brownian paths on the line.

7.2 THEOREM. For quasi-every path $b(\cdot)$ in W_0^1 ,

$$\dim \{t: b(t) = 0\} = \frac{1}{2}.$$

Proof. The same methods apply, for almost all ω and all τ . By the Hölder continuity in t of the local time $\alpha(\tau, 0, t)$ (Theorem 4.15), combined with the fact that $\alpha(\tau, 0, 1)$ is positive (Theorem 6.3), and Lemma 2.5,

$$h_3 - m([0, 1] \cap (b^\tau)^{-1}\{0\}) > 0$$

where
$$h_3(u) = (u |\log u|)^{\frac{1}{2}}.$$

Conversely, by Lemma 2.6 and the Hölder continuity of the Brownian sheet (Theorem 1.6),

$$h_4 - m([0, 1] \cap (b^\tau)^{-1}\{0\}) < \infty$$

where
$$h_4(u) = (u/|\log u|)^{\frac{1}{2}}.$$

□

7.3 Remark. The statement of Theorem 7.2 as a fact about the level set $w^{-1}(0)$ of a Brownian sheet $w: \mathbb{R}_+^2 \rightarrow \mathbb{R}$ is also quite interesting. The theorem says that every horizontal or vertical cross-section of the set $(w^{-1}(0)) \cap (0, \infty)^2$ has Hausdorff dimension $\frac{1}{2}$. It may be possible to generalise this statement to cross-sections associated with a larger class of straight lines or of curves which pass through an axis.

From Theorem 7.1 we now deduce the generalization to quasi-every Brownian path of results by Taylor (1966) ($d=2$) and Fristedt (1967) ($d=3$) on the dimension of the set of Brownian self-intersections.

7.4 THEOREM. Let $d=2$ or $d=3$. For quasi-every Brownian path $b(\cdot)$ in \mathbb{R}^d ,

$$\dim\{x: x = b(s)=b(t), \text{ some } (s,t) \in \Delta\} = 4-d.$$

To Prove Theorem 7.4, we need to show that for all τ the function $X^\tau(s,t) = b^\tau(t) - b^\tau(s)$ spreads out sets of dimension $2 - d/2$ in the plane to sets of dimension $4 - d$ in \mathbb{R}^d . We do this using a quasi-everywhere extension of an idea of Kaufman (1969):

7.5 LEMMA. Let $\tau_1 \geq 1$. With probability 1, there exists n_0 such that for all $n \geq n_0$, we have the following: if $\tau = m2^{-2n} \in [1, \tau_1]$ where $m \in \mathbb{Z}$, and B^* is a ball in \mathbb{R}^d of radius $n^{1/2} 2^{-n}$, then

$$b^\tau(k4^{-n}) \in B^* \text{ for at most } n^{3+\epsilon} \text{ values of } k \text{ in } \{1, \dots, 4^n\}.$$

Proof. Let $A_n^*(\tau)$ be the event that for some ball B^* of radius $n^{1/2} 2^{-n}$, $b^\tau(k4^{-n}) \in B^*$ for at most $n^{3+\epsilon}$ values of k in $\{1, \dots, 4^n\}$. By the estimate in Kaufman (1969),

$$P(A_n^*(\tau)) \leq (Cn)^n \left[\binom{n^{3+\epsilon}}{n} \right]^{-1} \quad (1 \leq \tau \leq \tau_1)$$

for some constant C depending only on τ_1 . Hence (for a different C),

$$P \cup \{A_n^*(m2^{-2n}): 1 \leq m2^{-2n} \leq \tau\} \leq (Cn)^n \left[\binom{n^{3+\epsilon}}{n} \right]^{-1}$$

and the result follows by Borel-Cantelli. \square

7.6 Remark. From Lemma 7.5 and the Hölder continuity of the Brownian sheet (Theorem 1.6), it follows that for large enough n , for all $\tau \in [1, \tau_1]$ and all balls B in \mathbb{R}^d of radius 2^{-n} ,

$(b^\tau)^{-1}(B)$ is contained in at most $n^{3+\epsilon}$ intervals of the form $[k4^{-n}, (k+1)4^{-n}]$. Following Geman, Horowitz and Rosen (1984), we introduce a result on functions on metric spaces:

7.7 LEMMA. Let (E_i, d_i) ($i=1,2$) and (F, d) be metric spaces.

Let $E_1 \times E_2$ have the metric given by the maximum of d_1 and d_2 .

Suppose X_i ($i=1,2$) are functions from E_i to F , and f and θ_i ($i=1,2$) are increasing continuous functions from \mathbb{R}_+ to \mathbb{R}_+ taking the value zero at zero. Suppose there exists $\delta > 0$ such that for every set B in F with diameter $d(B) < \delta$, $X_i^{-1}(B)$ can be covered by at most $\theta_i(d(B))$ sets in E_i of diameter $f(d(B))$. Set

$$A_0 = \{(x_1, x_2) \in E_1 \times E_2 : X_1(x_1) = X_2(x_2)\}$$

and $C = \{y \in F : y = X_1(x_1) = X_2(x_2), \text{ some } (x_1, x_2) \in A_0\}$

so that C is the set of confluences of X_1 and X_2 . Then for every Hausdorff measure function g ,

$$h - m(C) \geq g - m(A_0)$$

where

$$h(u) = g(f(u)) \theta_1(u) \theta_2(u).$$

Proof. Suppose $\{B_j, j \geq 1\}$ is a covering of C by sets B_j of diameter $d(B_j) \leq \epsilon$. Then $X_i^{-1}(B_j)$ can be covered by $n_i(j)$ sets of diameter $(f(d(B_j)))$, where $n_i(j) \leq \theta_i(d(B_j))$ ($i=1,2$). Hence $\{(x_1, x_2) : X_1(x_1) \in B_j, X_2(x_2) \in B_j\}$, a subset of $E_1 \times E_2$, can be covered by $n_1(j) n_2(j)$ balls in $E_1 \times E_2$ of diameter $f(d(B_j))$.

Hence

$$\sum_{j \geq 1} h(d(B_j)) \geq \sum_{j \geq 1} g(f(d(B_j))) n_1(j) n_2(j)$$

and the right hand side of the above is the sum of the values of g

on the diameters of an $f(\epsilon)$ -covering of A_0 . Allowing $\epsilon \rightarrow 0$ gives us the result. \square

7.8 Proof of Theorem 7.4. The following argument applies for almost all ω and all $\tau \in [1, \tau_1]$. First we obtain a lower bound on the dimension of the set of double points of b^τ . For each $\tau \in [1, \tau_1]$, it follows from (*) in the proof of Theorem 7.1 that there exists a square $I_1 \times I_2$ in the upper triangle Δ , where I_1 and I_2 are closed intervals, such that

$$h_1 - m \{(s, t) \in I_1 \times I_2 : b^\tau(s) = b^\tau(t)\} > 0$$

where, as in the proof of Theorem 7.1, $h_1(u) = u^{2-d/2} |\log u|^2$.

Apply Lemma 7.7 with $E_1 = I_1$, $E_2 = I_2$, $F = \mathbb{R}^d$,
 $X_1(t) = X_2(t) = b^\tau(t)$, $f(u) = u^2$, $\theta_1(u) = \theta_2(u) = |\log u|^{3+\epsilon}$
(using Remark 7.6). Thus

$$h_5 - m \{y \in \mathbb{R}^d : y = b^\tau(s) = b^\tau(t), \text{ some } (s, t) \in I_1 \times I_2\} > 0 \quad (1)$$

where $h_5(u) = u^{4-d} |\log u|^{10+\epsilon}$.

Conversely, by the Hölder continuity of the Brownian sheet (Theorem 1.6), the function $b^\tau(t)$ satisfies the Hölder condition

$$\sup\{|b^\tau(s) - b^\tau(s')|/g(|s-s'|) : \tau, s, s' \in [0, 1]\} < \infty$$

where $g(u) = (u |\log u|)^{1/2}$.

From the proof of Theorem 7.1, we also have an upper estimate (**) for the Hausdorff measure of the set $(X^\tau)^{-1}(0)$; it then follows easily from the Hölder continuity of $b^\tau(\cdot)$ (Theorem 1.6) that the image under b^τ of the projection onto the s -axis of this set (i.e. the set of double points of $b(\cdot)$) has finite Hausdorff h_6 -measure

relative to the measure function

$$h_6(u) = u^{4-d} |\log u|^{-2}.$$

This, together with (1), completes the proof. \square

7.9 Extension to more parameters. Theorems 7.1, 7.2 and 7.4 can be viewed as statements about the two-parameter Wiener process $(b^\tau(t): \tau \geq 0, t \geq 0)$ in \mathbb{R}^d . It is possible to extend these results to an $(N+1)$ -parameter Wiener process $(b^\tau(t): \tau \in \mathbb{R}_+^N, t \geq 0)$. The generalisations of Theorems 7.2 and 7.4 say that with probability 1, for all τ in $(0, \infty)^N$,

$$\dim \{t \in [0, 1]: b^\tau(t) = 0\} = \frac{1}{2} \quad (d=1)$$

and

$$\dim \{x \in \mathbb{R}^d: x = b^\tau(s) = b^\tau(t), \text{ some } s \neq t \text{ in } [0, 1]\} = 4-d \quad (d=2 \text{ or } d=3).$$

The appropriate modifications of the proofs in Chapters 3, 4, 6 and 7 are straightforward.

7.10 Some open problems. Taylor and Wendel (1966) found the exact measure function h_7 for the level set of W -almost every path $b(t)$ on the line; that is, they showed

$$0 < h_7 - m(b^{-1}\{0\} \cap [0, 1]) < \infty \quad W\text{-a.s.}$$

where
$$h_7(u) = u^{\frac{1}{2}} (\log |\log u|)^{\frac{1}{2}}.$$

Does the same result hold quasi-everywhere, or is there a positive probability that for some $\tau > 1$, the Hausdorff h_7 measure of $(b^\tau)^{-1}(0) \cap [0, 1]$ is zero or infinite? The proof of the a.e. result uses the fact that the local time at 0 up to time t satisfies the law of the iterated logarithm in t . We do not know whether this law holds for quasi-every path.

Similar questions may be asked about the exact measure function for the set of self-intersections of a Brownian path in \mathbb{R}^3 . In fact, as far as we know the question of the exact measure function of the set of self-intersections for almost all paths is still open (let alone the quasi-everywhere question). See Le Gall (1986) for a use of intersection local time in this context when $d = 2$.

8.1 Introduction. The work presented in this chapter arose in conjunction with the study of the self-intersections of a Brownian path in \mathbb{R}^3 as the path varies according to a Wiener (or Ornstein-Uhlenbeck) process in Wiener space. Consider the values of the path $b^\tau(t)$ at times $t = t_1, \dots, t_N$ (here N is large) forming a dissection of $[0,1]$. As τ varies these values $b^\tau(t_i)$ evolve by N Wiener processes (up to scalar multiples). Consider the separations

$$R_{ij}(\tau) = |b^\tau(t_i) - b^\tau(t_j)| \quad (1 \leq i < j \leq N)$$

between values of the process at different times in the dissection. These evolve according to $O(N^2)$ Bessel processes as τ varies.

The minimum separation of the path for pairs of times in the dissection then evolves, as τ varies, according to a stochastic process $X(\tau)$ given by

$$X(\tau) = \min\{R_{ij}(\tau) : 1 \leq i < j \leq N\}, \quad (*)$$

the (pointwise) minimum of $O(N^2)$ Bessel processes. To investigate those τ where the set of self-intersections of $b^\tau(\cdot)$ is most sparse, we might consider the maximum of $\{X(\tau), 0 \leq \tau \leq 1\}$ (and then refine the dissection so $N \rightarrow \infty$).

A similar question arises in the context of rates of chemical reaction, as discussed by Clifford, Green and Pilling (1984). Suppose that N molecules are created at the same place in a suspension fluid by incident radiation, and then perform independent Brownian motions. The inter-molecular distances (i.e. the distances

between the centres of molecules) then perform $O(N^2)$ Bessel processes, again denoted $R_{ij}(\tau)$, ($1 \leq i < j \leq N$). Suppose two molecules interact if their separation is less than some fixed amount $2r_0$ (the molecular diameter) after some threshold time τ_0 . Then the question of whether an interaction is taking place at time $\tau > \tau_0$ depends on whether the value of the minimal separation is less than $2r_0$. Hence we might again be interested in a process $X(\tau)$ given by (*), i.e. by the (pointwise) minimum of a family of N^2 Bessel processes. In particular, a reaction takes place between times τ_1 and τ_2 if

$$\text{Min}\{X(\tau): \tau \in [\tau_1, \tau_2]\} \leq 2r_0.$$

In both cases above our Bessel processes are highly dependent, which makes calculation difficult. Here we shall concentrate on the easier case of n independent Bessel processes (n large). A precedent for this simplification can be found in Clifford, Green and Pilling (1984), section 3.1.

Our work is closely related to that of Brown and Resnick (1977) who consider the process obtained by taking the maximum of a family of Brownian motions, pointwise in the time parameter. Brown and Resnick give as their motivation, "... certain extreme phenomena, such as river flows both in time of drought and flooding, may be such that the extreme behaviour over a time interval ... may be of interest." In such examples, the physical processes may well take values in \mathbb{R}_+ rather than \mathbb{R} , which provides some motivation for considering the Bessel process, as we do here, rather than Brownian

motion. We observe that the value at time t of a Bessel process lies in a different domain of attraction from the value at time t of Brownian motion.

There are three main sections to the work here. First we shall consider a process $M_n(t)$ whose value at t is given by the minimum of the values of n Bessel processes at time t . We examine the asymptotic distribution of the process $M_n(t)$, suitably re-normalised in space and time, as $n \rightarrow \infty$. Our result is a Bessel process analogue to Theorem 1 of Brown and Resnick (1977). We shall then discuss the maximal and minimal functions associated with $M_n(\cdot)$: i.e. we consider

$$\max \{M_n(t) : t_0 \leq t \leq t_2\}$$

and

$$\min \{M_n(t) : t_0 \leq t \leq t_2\},$$

re-normalised in space only. This is motivated by the discussion above.

8.2 Notation. Let d be a positive integer. Let (Ω, Σ, P) be a probability space and

$$(B(t))_{t \geq 0}, (B_1(t))_{t \geq 0}, (B_2(t))_{t \geq 0}, (B_3(t))_{t \geq 0}, \dots$$

be mutually independent Brownian motions in \mathbb{R}^d , starting at the origin, with underlying probability space (Ω, Σ, P) . Let

$R(t) = |B(t)|$ be the Euclidean norm of $B(t)$. Then $(R(t))_{t \geq 0}$ is a Bessel process of index d , starting at 0, which we denote a $Bes(d)$ process. Let $R_i(t) = |B_i(t)|$, $i = 1, 2, 3, \dots$

Define

$$M_n(t) = \min\{R_i(t) : 1 \leq i \leq n\}.$$

Define $c = c(d)$ by

$$\begin{aligned} c &= (\text{Volume of unit ball in } \mathbb{R}^d) / (2\pi)^{d/2} \\ &= \lim_{h \downarrow 0} P[M_n(t) \leq h] / (h^d t^{1/2}) . \\ &= \sup_{h \geq 0} P[M_n(t) \leq h] / (h^d t^{1/2}) . \end{aligned}$$

8.3 Remark. For $d \geq 2$, a Bes (d) process may alternatively be characterised by its infinitesimal generator A , given by

$$Af(r) = \left(\frac{1}{2}\right) f''(r) + \frac{d-1}{2r} f'(r) \quad (r \geq 0).$$

From the generator, we may express a Bes (d) process $R(t)$ by

$$R(t) = w(t) + \frac{1}{2} (d-1) \int_0^t (1/R(s)) ds,$$

where $w(t)$ is a Brownian motion on the line. With this formulation, the results in this chapter carry over to non-integer d , although we present our proofs in terms of the Bessel process as the modulus of a Brownian motion in Euclidean space.

8.4 PROPOSITION. If $0 < t_1 < t_2 < \dots < t_J < \infty$, and

$\alpha_j \in [0, \infty)$, $1 \leq j \leq J$, then

$$\lim_{n \rightarrow \infty} P[n^{1/d} t_j^{-1/2} M_n(t_j) > \alpha_j, 1 \leq j \leq J] = \exp(-\sum_{j=1}^J (\alpha_j)^d / c). \quad (**)$$

The case $J = 1$ of Proposition 8.4 says that $n^{1/d} t^{-1/2} M_n(t)$ converges in law to the distribution function

$$G(x) = 1 - \exp(-x^d / c) \quad (x \geq 0)$$

as $n \rightarrow \infty$; i.e. $t^{-1/2} R(t)$ lies in the domain of attraction of G . This is a standard type of result in the theory of extremal order statistics (see for example Galambos (1978), Corollary 1.3.2).

The result (**) still holds if $n^{1/d}$ is replaced by any sequence a_n such that $a_n = n^{1/d}(1+o(1))$.

8.5 A re-normalisation of the time parameter. Given $t_0 \geq 0$, and given any sequence $\zeta = (\zeta_n, n \geq 1)$ of positive reals, we re-scale the time-parameter by defining

$$\begin{aligned}\tau_n(t) &= \tau_n(t_0, \zeta, t) = t_0 + \zeta_n n^{-2/d} t. \\ \bar{M}_n(t) &= M_n(\tau_n(t)).\end{aligned}$$

Our next two results assert that for fixed $t_0 > 0$, the range of t over which $n^{1/d} M_n(t_0)$ influences $n^{1/d} M_n(t)$ is of size proportional to $n^{-2/d}$. Over ranges shrinking more slowly than $n^{-2/d}$, the finite-dimensional distributions of $n^{1/d} M_n$ converge in law to those of a process whose values at different times are mutually independent. Over ranges shrinking faster than $n^{-2/d}$, the limiting distribution of $n^{1/d} M_n(\cdot)$ is that of a constant process.

We note here that the scaling property of Brownian motion implies that $((\lambda t)^{-1/2} M_n(\lambda t), t \geq 0)$ has the same finite-dimensional distributions as $(M_n(t), t \geq 0)$. This implies that in the case $t_0 = 0$, $((\tau_n(t))^{-1/2} \bar{M}_n(t), t \geq 0)$ has the same distributions as $(t^{-1/2} M_n(t), t \geq 0)$, irrespective of the sequence ζ .

8.6 PROPOSITION. Suppose $t_0 > 0$, and $\zeta_n \rightarrow \infty$ ($n \rightarrow \infty$). Let

$0 < t_1 < t_2 < \dots < t_J < \infty$, and $\alpha_j \in [0, \infty)$, ($1 \leq j \leq J$). Then

$$\lim_{n \rightarrow \infty} P\left[\bigcap_{j=1}^J \{n^{1/d} \bar{M}_n(t_j)(\tau_n(t_j))^{-1/2} > \alpha_j\}\right] = \exp\left(-\sum_{j=1}^J (\alpha_j)^d / c\right).$$

8.7 PROPOSITION. Suppose $t_0 > 0$, and $\zeta_n \rightarrow 0$ ($n \rightarrow \infty$). Let $(Y_\infty(t))_{t \geq 0}$ be a constant-in-time process with distribution function G , i.e.

$$P[Y_\infty(t) = Y_\infty(0), t \geq 0] = 1,$$

and $P[Y_\infty(0) \geq \alpha] = \exp(-\alpha^d/c)$ ($\alpha \geq 0$).

Then the processes $t_0^{-1/2} n^{1/d} \bar{M}_n(\cdot)$ converge narrowly ("weakly") in $C[0, \infty)$ to $Y_\infty(\cdot)$ as $n \rightarrow \infty$.

8.8 Remarks. Proposition 8.4 is a special case of Proposition 8.6.

Throughout this chapter, $C[0, \infty)$ is given the topology of uniform convergence on bounded intervals, which is metrisable. In Proposition 8.7, we could work on $C(-\infty, \infty)$ if we wished, since the domain of definition of \bar{M}_n is $\{t: \tau_n(t) > 0\}$, which expands to $(-\infty, \infty)$ as $n \rightarrow \infty$. For proofs of Propositions 8.6 and 8.7, see Penrose (1988b); the proof of Proposition 8.6 is elementary, while Proposition 8.7 may be proved in much the same way as Theorem 8.16 below.

We now consider the critical case of the limiting distribution of $\bar{M}_n(\cdot)$ when $\zeta_n = 1$, for all n , using the methods of Brown and Resnick (1977). We shall show that \bar{M}_n converges narrowly to a process obtained by taking the pointwise minimum of the moduli of a family of paths in \mathbb{R}^d , with initial values given by a Poisson process in \mathbb{R}^d , and subsequent evolution given by independent Brownian motions in \mathbb{R}^d . To make this description precise we recall

the following generalisation of the familiar Poisson process on the line:

8.9 The Poisson random measure. Let (E, Σ, m) be a σ -finite measure space. A *Poisson random measure* on E with *intensity* m is a random, integer valued, countably additive set function

$$\mu: \Sigma \rightarrow Z_+$$

(with some underlying probability space (Ω, \mathcal{F}, P)), such that if $\{E_i, i = 1, \dots, k\}$ are disjoint sets in Σ and $\{n_i, i \geq 1\}$ are integers,

$$P\{ \mu(E_i) = n_i, 1 \leq i \leq k \} = \prod_{j=1}^k p(m(E_j), n_j)$$

where $p(\lambda, \cdot)$ is the poisson probability function with parameter λ :

$$p(\lambda, r) = e^{-\lambda} \lambda^r / r! \quad 0 < \lambda < \infty, r < \infty$$

$$p(0, 0) = p(\infty, \infty) = 1.$$

$$p(\lambda, r) = 0 \quad \text{otherwise}$$

A constructive proof of the existence of Poisson random measures can be found in Brown (1970), page 1939.

8.10 The Limit Process. Fix $t_0 > 0$. Let π be a homogeneous Poisson random measure on \mathbb{R}^d with intensity

$$m(dx) = t_0^{d/2} (c\pi_d)^{-1} dx.$$

Here $c = c(d)$ is the constant introduced earlier (in section 8.2) and π_d is the volume of the unit ball in \mathbb{R}^d . Let $(\Omega_1, \mathcal{F}_1, P_1)$ be the underlying probability space of π . Let $(x_i(\pi), i = 1, 2, 3, \dots)$ be the atoms of π taken in order of increasing Euclidean norm (with

probability 1, the $|x_i|$ are distinct).

Let $(B_1(\cdot), B_2(\cdot), B_3(\cdot), \dots)$ be a family of independent Brownian motions in \mathbb{R}^d , each starting at zero, with underlying probability space $(\Omega_2, \mathcal{F}_2, P_2)$ (an infinite product of d -dimensional Wiener space). Then we can define a random point measure μ on $C[0, \infty)$ over the product of the probability spaces (Ω_1, P_1) and (Ω_2, P_2) , by taking μ to have an atom at each $X(\cdot) \in C[0, \infty)$ of the form

$$X(t) = |x_i(\pi) + B_i(t)|, \quad t \geq 0$$

for some $i \in \{1, 2, 3, \dots\}$.

Let the $(t_0$ -dependent) random function $Y(\cdot)$ be the pointwise minimum of the atoms (considered as functions in $C[0, \infty)$) of μ , i.e.

$$Y(t) = \min \{X(t) : X \text{ an atom of } \mu\}.$$

We shall see later that Y is (almost surely) an element of $C[0, \infty)$. Y is our limiting $C[0, \infty)$ -valued random variable.

8.11 Remarks. The limit process $Y(\cdot)$ can equivalently be formulated as the minimum of a family of $\text{Bes}(d)$ Processes on \mathbb{R}_+ , whose initial positions are given by a Poisson process with intensity $m'[0, x^d] = (t^{d/2}/c) x^d$, and whose subsequent evolutions are mutually independent. This formulation also makes sense for non-integer d .

The idea of a set of particles moving in \mathbb{R}^d , with initial position given by a homogeneous poisson process and subsequent motion given by independent Brownian motions, goes back at least to Doob

(1953), page 404. Since the homogeneous measure m on \mathbb{R}^d is invariant under the transition probabilities of Brownian motion, the particles are in statistical equilibrium.

By Theorem 1 of Brown (1970), the random measure μ , in terms of which the limit process Y is defined, is in fact a Poisson random measure on $C[0, \infty)$ (with its Borel σ -algebra), with intensity m_∞ given by

$$m_\infty(A) = (m \times W) \{ (x, b(\cdot)) : |x+b(\cdot)| \in A \} \quad (A \in \mathcal{B}C[0, \infty))$$

where W is d -dimensional Wiener measure.

Note that for $t \geq 0$ and $a > 0$,

$$m_\infty \{ f(\cdot) \in C[0, \infty) : f(t) \leq a \} = a^d/c.$$

Hence $P(Y(t) \leq a) \leq \exp(-a^d/c).$

It is routine to check that the limit process $Y(t)$ has stationary transition probabilities.

8.12 LEMMA. The process $Y(t)$ is continuous in t , almost surely.

Proof. Given $r > 0$ and $t \in (0, \infty)$, we have

$$\begin{aligned} M_\infty \{ (x, b(\cdot)) : |x+b(s)| < r, \text{ some } s \in [0, t] \} \\ = \text{const.} \times \int_0^\infty q(x, r, t) x^{d-1} dx \end{aligned} \quad (1)$$

where $q(x, r, t)$ is the probability that a $Bes(d)$ process, starting at x , goes below r during the time-interval $(0, t)$.

Hence the right hand side of (1) is finite, so $(Y(s), 0 \leq s \leq t)$ is in fact the pointwise minimum of a finite (random) number of continuous functions. Hence $Y(\cdot)$ is continuous almost surely.

8.13 LEMMA. As before, let $\bar{M}_n(t) = M_n(t_0 + n^{-2/d} t)$, where $M_n(\cdot)$ is a pointwise minimum of the independent $\text{Bes}(d)$ processes $B_i(\cdot), 1 \leq i \leq n$. Then the finite-dimensional distributions of the processes

$$Y_n(\cdot) = n^{1/d} \bar{M}_n(\cdot)$$

converge in law to those of the limit process Y described above.

Proof. Let A be any finite subset of $[0, \infty)$. Without loss of generality $0 \in A$. We are required to prove the assertion that

$$(Y_n(t), t \in A) \xrightarrow{L} (Y(t), t \in A) \quad (n \rightarrow \infty) \quad (1)$$

where \xrightarrow{L} denotes convergence in law of \mathbb{R}^A -valued random variables.

Let S_A be the set of point measures on \mathbb{R}^A with the metrisable topology such that for ν_n and ν in S , ν_n converges to ν if and only if for all $f \in C_0(\mathbb{R}^A)$, $\int f d\nu_n \rightarrow \int f d\nu$. An alternative description of this topology is that $\nu_n \rightarrow \nu$ if and only if for all bounded hypercubes $Q \subset \mathbb{R}^A$ (viewed as Euclidean space) with no atoms of ν in ∂Q , $\nu_n(Q) \rightarrow \nu(Q)$.

Define $h_A: S_A \rightarrow \mathbb{R}_+^A$ by

$$(h_A(\nu))(t) = \min \{X(t) : X \text{ an atom of } \nu\} \quad t \in \bar{A}.$$

Then h_A is continuous.

Let π_A be the projection from $\{\text{point measures on } C[0, \infty)\}$ onto S_A induced by the obvious projection from $C[0, \infty)$ onto \mathbb{R}^A .

For $n \geq 1$, let μ_n be the (random) point measure on $C[0, \infty)$ with atoms at each of the functions $|n^{1/d} B_i(t_0 + n^{-2/d}(\cdot))|$ ($1 \leq i \leq n$), considered as points in $C[0, \infty)$.

Assertion (1) says that the \mathbb{R}^A -valued random variables $h_A(\pi_A(\mu_n))$ converge in law to $h_A(\pi_A(\mu))$ where μ is the Poisson random measure on $C[0, \infty)$ described in section 8.10. By the continuous mapping theorem (Billingsley (1968), Theorem 5.1) it suffices to prove that the S_A -valued random variables $\pi_A(\mu_n)$ converge weakly to $\pi_A(\mu)$.

The measure $\pi_{\{0\}}(\mu)$ is a Poisson random measure on \mathbb{R} with intensity given by $m'[0, x) = (t_0^{d/2}/c) x^d$. By the one-dimensional case of Proposition 8.4, $Y_n(0) = n^{1/d} M_n(t_0)$ converges in law to the limit distribution

$$G'(x) = 1 - \exp(-m'(0, x)) \quad (x \geq 0)$$

where $m'(0, x) = (xt_0^{d/2})^d/c$ as before. Now by one-dimensional extreme value theory, the point measures $\pi_{\{0\}}(\mu_n)$ on \mathbb{R} with atoms at $\{n^{1/d} R_i(t_0), 1 \leq i \leq n\}$ converge weakly to a Poisson random measure on \mathbb{R} with intensity m' , the same as the intensity of the Poisson random measure $\pi_{\{0\}}(\mu)$. Indeed, by Theorem 1 of Resnick (1975), there exist random point measures ρ_n and ρ on \mathbb{R}_+ such that ρ_n and ρ are equal in law to $\pi_{\{0\}}(\mu_n)$ and $\pi_{\{0\}}(\mu)$; respectively, while ρ_n converges almost surely, in the topology on $S_{\{0\}}$ given above, to ρ .

Let \underline{e} be an arbitrary unit vector in \mathbb{R}^d . Let $w_1(\cdot), w_2(\cdot), \dots$ be a sequence of independent Brownian motions in \mathbb{R}^d . Define the random point measures μ'_n and μ' on \mathbb{R}^A as follows:

μ'_n has an atom at $(X(t), t \in A)$ if and only if for some $i \in \{1, \dots, n\}$, $X(0)$ is the i^{th} order statistic of the atoms of ρ_n (taken in increasing order), and

$$X(t) = |X(0) \underline{e} + w_i(t)| \quad t \in A.$$

μ' has an atom at $(X(t), t \in A)$ if and only if for some $i \geq 0$, $X(0)$ is the i^{th} order statistic of the atoms of ρ , and

$$X(t) = |X(0) \underline{e} + w_i(t)| \quad t \in A.$$

By the rotational invariance and other standard properties of Brownian motion in \mathbb{R}^d , together with the equality of the distributions of the random measures ρ_n and $\pi_{\{0\}}(\mu_n)$, we have equality in distribution of μ'_n and $\pi_A(\mu_n)$. Also, since the random measures ρ and $\pi_{\{0\}}(\mu)$ have the same distribution, μ' is equal in distribution to $\pi_A(\mu)$.

The almost sure convergence of ρ_n to ρ implies that for each i , the i^{th} order statistic of the atoms of ρ_n converges almost surely to the i^{th} order statistic of the atoms of ρ . Since the same sequence of Brownian paths $(w_i(\cdot), i \geq 1)$ is used for the construction of the atoms of μ'_n as for μ' , it follows that μ'_n converges almost surely to μ' (in the topology on S_A described above). Hence $\pi_A(\mu_n)$ converges in distribution to $\pi_A(\mu)$, and assertion (1) is proved. \square

To deduce narrow convergence of the laws of $n^{1/d} \bar{M}_n(\cdot)$, we need a tightness result. A set of probability measures $\{P_\alpha, \alpha \in A\}$ on a metric space E is said to be *tight* if for all $\epsilon > 0$, there exists compact $K \subset E$ with $P_\alpha(K) > 1 - \epsilon$, all $\alpha \in A$.

8.14 LEMMA. The laws of the Processes $Y_n(\cdot)$ given by

$$Y_n(t) = n^{1/d} \bar{M}_n(t) = n^{1/d} M_n(\tau_n(t))$$

(where $\tau_n(t) = t_0 + n^{-2/d}t$) form a tight family of probability measures on $C[0, \infty)$.

Proof. By Theorem (8.3) of Billingsley (1968), it suffices to show that for $0 < t_1 < \infty$, if $\epsilon > 0$ then for some $K > 0$, $n_0 \geq 1$ and $\delta \in (0, 1)$,

$$(i) \quad P[Y_n(0) \geq K] < \epsilon, \quad \text{all } n \geq 1.$$

$$(ii) \quad \delta^{-1} P[\max\{|Y_n(s) - Y_n(t)| : t \leq s \leq t + \delta\} > \epsilon] < \epsilon$$

$$(\text{all } n \geq n_0, t \in [0, t_1]).$$

Condition (i) holds for large K , since $Y_n(0)$ converges in distribution to a real-valued random variable. It remains to prove (ii).

Fix t . Let i^* be chosen to minimise $R_i(\tau_n(t))$ over $1 \leq i \leq n$. Suppose that $|Y_n(s) - Y_n(t)| > \epsilon$ for some $s \in [t, t + \delta]$. Then either

$$n^{1/d} |R_{i^*}(\tau_n(s)) - R_{i^*}(\tau_n(t))| > \epsilon, \quad \text{some } s \in [t, t + \delta],$$

or for some $j \neq i^*$, j takes over from i^* as index of the minimal Bessel process somewhere in the interval $[\tau_n(t), \tau_n(t + \delta)]$. Hence, either

$$Y_n(t) > \delta^{-1}$$

or

$$Y_n(t) \leq \delta^{-1}, \text{ and } n^{1/d} |R_{i^*}(\tau_n(s)) - R_{i^*}(\tau_n(t))| > \epsilon, \quad \text{some } s \in [t, t + \delta]$$

or

$$Y_n(t) \leq \delta^{-1}, \text{ and } n^{1/d} R_j(\tau_n(s)) < n^{1/d} R_{i^*}(\tau_n(t)) - \epsilon,$$

$$\text{some } j \in \{1, \dots, n\} \setminus \{i^*\}, s \in [t, t + \delta].$$

We shall call these three events A_n , B_n and C_n respectively. To obtain tightness condition (ii), it suffices to show that if δ is small enough and n is large enough, then

$$\max\{\delta^{-1}P(A_n), \delta^{-1}P(B_n), \delta^{-1}P(C_n)\} \leq \epsilon/3, \quad t \in [0, t_1].$$

First we consider A_n . By the definition of Y_n ,

$$\begin{aligned} \delta^{-1}P(A_n) &= \delta^{-1} P[M_n(\tau_n(t)) > \delta^{-1}] \\ &\rightarrow \exp(-\delta^{-d} t_0^{d/2}) \quad (n \rightarrow \infty), \end{aligned}$$

locally uniformly in t . Hence we can find n_0 such that for $t \in [0, t_1]$ and $n \geq n_0$ and all small δ ,

$$\delta^{-1} P(A_n) < \epsilon/3. \quad (1)$$

Now consider B_n . By the definition of the event B_n we have

$$B_n \subset \bigcup_{s \in [t, t+\delta]} \{|B_{i^*}(\tau_n(s)) - B_{i^*}(\tau_n(t))| > \epsilon n^{-1/d}\}.$$

Define $R^*(t) = \max\{R(s), 0 \leq s \leq t\}$. Applying the independence of Brownian increments, the Brownian scaling property and the fact that $\tau_n(s) - \tau_n(t) = n^{-2/d}(s-t)$, we have

$$\begin{aligned} P(B_n) &\leq P[R^*(1) > \epsilon \delta^{-1/2}] \\ &= o(\delta) \quad (\delta \rightarrow 0) \quad (\text{uniformly in } n, t). \end{aligned} \quad (2)$$

Now consider C_n . The event C_n is contained in the union over all distinct i and j in $\{1, \dots, n\}$ of events $E_{n,i,j} \cap F_{n,i,j}$, where (writing R_i for $R_i(\tau_n(t))$ and similarly for R_j)

$$E_{n,i,j} = \{n^{1/d} R_i < \delta^{-1}\} \cap \{n^{1/d} R_j \geq n^{1/d} R_i\},$$

$$F_{n,i,j} = \{|B_j(\tau_n(s)) - B_j(\tau_n(t))| > R_j - R_i + n^{-1/d} \epsilon, \text{ some } s \in [t, t+\delta]\}.$$

By the Brownian scaling property,

$$P[F_{n,i,j} | \Sigma(R_i, R_j)] = P[R^*(1) \geq \delta^{-1/2} (n^{1/d}(R_j - R_i) + \epsilon)].$$

Hence

$$P[E_{n,i,j} \cap F_{n,i,j}] \leq \int_{x=0}^{\delta^{-1}} \int_{y=x}^{\infty} P[R^*(1) \geq \delta^{-1/2}(y-x+\epsilon)] P[n^{1/d}R_i \in dx, n^{1/d}R_j \in dy]. \quad (3)$$

Define $(f(\tau, x); x \geq 0, \tau > 0)$ so that f is continuous in x and

$$P[R_1(\tau) \in dx] = f(\tau, x) x^{d-1} dx,$$

$$\text{so } P[n^{1/d}R_1(\tau) \in dx] = f(\tau, n^{-1/d}x) x^{d-1} n^{-1} dx.$$

Then $f(\tau, \cdot)$ is bounded in \mathbb{R}_+ . In fact we can find a uniform upper bound M for $\{f(\tau_n(t), x) : n \geq 1, x \geq 0, t \in [0, t_1]\}$. Hence we have for $n \geq 1$,

$$P[n^{1/d}R(\tau_n(t)) \in dx] \leq M x^{d-1} n^{-1} dx$$

and similarly for R_i and R_j . Applying this to (3), we have

$$P(E_{n,i,j} \cap F_{n,i,j}) \leq \int_{x=0}^{\delta^{-1}} \int_{y=x}^{\infty} P[R^*(1) \geq \delta^{-1/2}(y-x+\epsilon)] M^2 x^{d-1} y^{d-1} n^{-2} dy dx.$$

Since the number of pairs of distinct i, j in $\{1, \dots, n\}$ is less than n^2 ,

$$\begin{aligned} P(C_n) &\leq \sum_{i \neq j} P(E_{n,i,j} \cap F_{n,i,j}) \\ &\leq M^2 \left[\int_0^{\delta^{-1}} x^{d-1} dx \right] \left[\int_{u=0}^{\infty} P[R^*(1) \geq \delta^{-1/2}(u+\epsilon)] (u+\delta^{-1})^{d-1} du \right]. \end{aligned}$$

Here we have changed variable to $u = y - x$. We split the second integral into $u < \delta^{-1}$ and $u > \delta^{-1}$; since

$$(u+\delta^{-1})^{d-1} \leq 2^{d-1} (\text{Max}(u, \delta^{-1}))^{d-1},$$

$P(C_n)$ is at most a constant times

$$\begin{aligned} & \delta^{-d} \left[\int_0^{\delta^{-1}} P[R^*(1) \geq \delta^{-1/2} u] \delta^{1-d} du + \int_{\delta^{-1}}^{\infty} P[R^*(1) \geq \delta^{-1/2} u] u^{d-1} du \right] \\ &= \delta^{-2d} P[R^*(1) \geq \delta^{-1/2}] + \delta^{-d/2} \int_{\delta^{-3/2}}^{\infty} P[R^*(1) \geq v] v^{d-1} dv \\ &= o(\delta) \quad (\delta \rightarrow 0), \end{aligned} \tag{4}$$

because $P[R^*(1) \geq x]$ has an exponentially decaying factor as $x \rightarrow \infty$.

The estimates (1), (2) and (4) together imply tightness condition (ii), as required. \square

Combining the last three lemmas, we have by Theorem (8.1) of Billingsley (1968) the following:

8.15 THEOREM. The laws of the processes $Y_n(\cdot)$ converge narrowly in $C[0, \infty)$ to that of the process $Y(\cdot)$.

8.16 Remarks. If, instead of the minimum of $\{R_i(t), 1 \leq i \leq n\}$, we consider the k^{th} order statistic (taken in increasing order), for some fixed $k \in \mathbb{Z}_+$, the above methods still apply.

As in Brown and Resnick (1977), we may consider instead of Brownian motion in \mathbb{R}^d , an Ornstein-Uhlenbeck process in \mathbb{R}^d . Let $(X(t), t \geq 0)$ be a standard Ornstein-Uhlenbeck process in \mathbb{R}^d , with initial distribution given by a standard normal. Then using the representation

$$X(t) - X(0) = w(t) - \left(\frac{1}{2}\right) \int_0^t X(s) ds,$$

(where $w(t)$ is Brownian motion), it is not hard to show that the

processes

$$n^{1/d} (X(n^{-2/d}t) - X(0)), \quad t \geq 0$$

converge narrowly, as $n \rightarrow \infty$, to Brownian motion in \mathbb{R}^d , i.e. the drift term of the Ornstein-Uhlenbeck process becomes unimportant under re-scaling. Hence, by the proof of Theorem 8.15, if $X_i(\cdot)$, ($i = 1, 2, 3, \dots$) are independent copies of the process $X(\cdot)$, then the laws of the processes

$$n^{1/d} \min \{|X_i(n^{-2/d}t)|, 1 \leq i \leq n\}$$

converge narrowly in $C[0, \infty)$ to that of the limit process Y (with $t_0 = 1$) described above.

8.17 The Maximal function of M_n . By contrast with the above

results, which describe the local interactions of the process $n^{1/d}M_n(\cdot)$, we now consider the global maxima of $M_n(\cdot)$. For any process $(X(\tau), \tau \geq 0)$ and $0 \leq s < t < \infty$, define the maximal function

$$X^*(s, t) = \max\{X(\tau), s \leq \tau \leq t\}.$$

Informally, the following theorem asserts that the distribution of the maximal function $M_n^*(0, t)$ converges to zero in proportion to $(n^{-1} \log n)^{1/d}$ (whereas the distribution of $M_n(t)$ converges to zero in proportion to $n^{-1/d}$).

Define the process $(V_n(t), t \geq 0)$ by

$$V_n(t) = (n/\log n)^{1/d} M_n^*(0, t)$$

and the deterministic process $(V_\infty(t), t \geq 0)$ by

$$V_\infty(t) = (2/cd)^{1/d} t^{k/2}, \quad t \geq 0.$$

Here, as before $c = c(d) = \lim_{h \downarrow 0} P[R(t) < ht^{k/2}]/h^d$

$$= \sup_{h \geq 0} P[R(t) < ht^{k/2}]/h^d.$$

8.18 THEOREM. The processes $(V_n(t))_{t \geq 0}$ converge narrowly in $C[0, \infty)$, as $n \rightarrow \infty$, to the deterministic process $(V_\infty(t))_{t \geq 0}$.

Proof. We shall show that for $t > 0$ and $\alpha \geq 0$,

$$\lim_{n \rightarrow \infty} P[V_n(t) > \alpha t^{1/2}] = \begin{cases} 1 & \text{if } \alpha < (2/cd)^{1/d} \\ 0 & \text{if } \alpha > (2/cd)^{1/d} \end{cases}. \quad (***)$$

This implies that for all $t \geq 0$, $V_n(t)$ converges in distribution to $(2/cd)^d t^{1/2} = V_\infty(t)$. Since $V_\infty(\cdot)$ is a deterministic process, this implies that the finite dimensional distributions of $V_n(\cdot)$ converge to those of $V_\infty(\cdot)$. Since for each n , $V_n(t)$ is monotone increasing in t , and the limit process $V_\infty(\cdot)$ is continuous, the narrow convergence in $C[0, \infty)$ of V_n to V_∞ follows.

We now prove (***), starting with the case $\alpha > (2/cd)^{1/d}$, so $c \alpha^d > 2/d$. Set r_n such that

$$\{V_n(t) > \alpha t^{1/2}\} = \{M_n^*(0, t) > r_n\},$$

$$\text{i.e.} \quad r_n = \alpha t^{1/2} (n^{-1} \log n)^{1/d}.$$

Choose β, γ such that $c \alpha^d > \beta > \gamma > 2/d$. Subdivide the time-interval $[0, t]$ into subintervals of length $n^{-\beta}$, by setting

$$t_j = j n^{-\beta}, \quad 0 \leq j \leq [t n^\beta] + 1.$$

Now, if $M_n(\cdot)$ exceeds r_n in the interval $[0, t]$, then it must exceed r_n in one of the subintervals, and the $R_i(\cdot)$ must all exceed r_n in this subinterval, i.e.

$$\{M_n^*(0, t) > r_n\} \subset \bigcup_{j=0}^{[t n^\beta]} \bigcap_{i=1}^n \{R_i^*(t_j, t_{j+1}) > r_n\}. \quad (1)$$

The triangle inequality in \mathbb{R}^d implies that for $s \in [t_j, t_{j+1}]$, $|B(s)| \leq r_n$ if $|B(t_j)| \leq r_n n^{-\gamma/2}$ and $|B(s) - B(t_j)| \leq n^{-\gamma/2}$.

Hence, since Brownian motion has stationary independent increments,

$$\begin{aligned}
 P[R^*(t_j, t_{j+1}) \leq r_n] &\geq P[R(t_j) \leq r_n n^{-\gamma/2}] \times \\
 &\quad \times P[\max_{t_j \leq t \leq t_{j+1}} |B(t) - B(t_j)| \leq n^{-\gamma/2}] \\
 &= P[R(t_j) \leq r_n n^{-\gamma/2}] \cdot P[R^*(n^{-\beta}) \leq n^{-\gamma/2}] \\
 &= P[R(t_j) \leq r_n n^{-\gamma/2}] \cdot P[R^*(1) \leq n^{(\beta-\gamma)/2}] \quad (2)
 \end{aligned}$$

where the last equality follows from the scaling property of Brownian motion.

$$\begin{aligned}
 \text{Now, } n^{-\gamma/2}/r_n &\rightarrow 0 \text{ as } n \rightarrow \infty, \text{ so for } t_j \leq t, \\
 P[R(t_j) \leq r_n n^{-\gamma/2}] &\geq P[R(t) \leq r_n n^{-\gamma/2}] \\
 &\geq c r_n^d \cdot t^{-d/2} (1+o(1)).
 \end{aligned}$$

Also, if we define q_n by

$$q_n = P[R^*(1) \leq n^{(\beta-\gamma)/2}],$$

then $q_n \rightarrow 1$ as $n \rightarrow \infty$. Using these facts in (2),

$$\begin{aligned}
 P[R^*(t_j, t_{j+1}) \leq r_n] &\geq c r_n^d \cdot t^{-d/2} q_n (1+o(1)) \\
 &= c \alpha^d \cdot (n^{-1} \log n) (1+o(1)) \quad (n \rightarrow \infty).
 \end{aligned}$$

Applying this result to the probabilities of the sets in (1), we have

$$\begin{aligned}
 P[M_n^*(0, t) > r_n] &\leq \sum_{j=0}^{[tn^\beta]} (1 - P[R^*(t_j, t_{j+1}) \leq r_n])^n \\
 &\leq ([tn^\beta] + 1) (1 - c \alpha^d \cdot n^{-1} (\log n) (1+o(1)))^n \\
 &= tn^{\beta - c \alpha^d} (1+o(1)) \\
 &\rightarrow 0 \quad (n \rightarrow \infty).
 \end{aligned}$$

Now set $\alpha < (2/cd)^{1/d}$. Choose θ so that $c \alpha^d < \theta < 2/d$.

As before, set

$$r_n = \alpha t^{\frac{k}{2}} (n^{-1} \log n)^{1/d},$$

so $\{V_n(t) > \alpha t^{\frac{k}{2}}\} = \{M_n^*(0, t) > r_n\}$.

It suffices to prove that $P[M_n^*(t-\delta, t) > r_n] \rightarrow 1$ as $n \rightarrow \infty$, for any positive δ so small that $(t/(t-\delta))^{d/2} < \theta/c\alpha^d$.

Again dissect the time-interval, this time by

$$t_j = jn^{-\theta}, \quad j = 0, 1, 2, 3, \dots$$

Define the random integer

$$N_n = \#\{j : t-\delta \leq t_j < t, \quad M_n(t_j) > r_n\}.$$

Clearly,

$$P[M_n^*(t-\delta, t) > r_n] \geq P(N_n > 0).$$

Following Hawkes (1978), we estimate $P[N_n > 0]$ from below as follows; by Cauchy-Schwarz,

$$E(N_n) \leq (E(N_n^2))^{1/2} (P(N_n > 0))^{1/2}.$$

Hence,

$$P(N_n > 0) \geq (E(N_n))^2 / E(N_n^2). \quad (3)$$

Now

$$\begin{aligned} E(N_n) &= \sum_{j: t-\delta \leq t_j < t} P[M_n(t_j) > r_n] \\ &\geq [\delta n^\theta] (P[R(t-\delta) > r_n])^n. \end{aligned}$$

By the definition of c and r_n , we have for all $s > 0$,

$$\begin{aligned} P[R(s) \leq r_n] &\leq c r_n^d s^{-d/2} \\ &= c \alpha^d \cdot (n^{-1} \log n) (t/s)^{d/2}. \end{aligned} \quad (4)$$

Setting $s = t - \delta$,

$$\begin{aligned} E(N_n) &\geq [\delta n^\theta] \{1 - c \alpha^d (n^{-1} \log n) (t/(t-\delta))^{d/2}\}^n \\ &= [\delta n^\theta] \exp\{-c \alpha^d (t/(t-\delta))^{d/2} (\log n) (1+o(1))\} \\ &= \delta n^{\theta - c \alpha^d (t/(t-\delta))^{d/2} (1+o(1))} \end{aligned}$$

$\rightarrow \infty$ ($n \rightarrow \infty$), by choice of δ small enough.

Also,

$$(E(N_n))^2 = \sum_j \sum_k P[M_n(t_j) > r_n] \cdot P[M_n(t_k) > r_n] . \quad (5)$$

Here and below, the summation is over those j and k such that

t_j and t_k are in $[t-\delta, t]$. Now consider the second moment of N_n .

$$\begin{aligned} E(N_n^2) &= \sum_j \sum_k P[M_n(t_j) > r_n, M_n(t_k) > r_n] \\ &= E(N_n) + 2 \sum_{j < k} P[M_n(t_j) > r_n, M_n(t_k) > r_n] . \end{aligned} \quad (6)$$

We compare the terms in this last summation to the corresponding terms in the summation (5) for $(E(N_n))^2$.

For $t - \delta \leq t_j < t_k < t$,

$$\begin{aligned} P[M_n(t_j) > r_n, M_n(t_k) > r_n] &= (P[R(t_j) > r_n, R(t_k) > r_n])^n \\ &= (1 - P[R(t_j) \leq r_n] - P[R(t_k) \leq r_n] + P[R(t_j) \leq r_n, R(t_k) \leq r_n])^n . \end{aligned} \quad (7)$$

But clearly

$$P[R(t_j) \leq r_n, R(t_k) \leq r_n] \leq P[R(t_j) \leq r_n] \cdot P[R(t_k - t_j) \leq r_n] .$$

For the j, k under consideration, $t_j > t - \delta$ and $t_k - t_j \geq n^{-\theta}$, so that

$$P[R(t_j) \leq r_n, R(t_k) \leq r_n] \leq P[R(t - \delta) \leq r_n] P[R(n^{-\theta}) \leq r_n]$$

Applying the estimate (4), we obtain

$$P[R(t_j) \leq r_n, R(t_k) \leq r_n] \leq (t/(t-\delta))^{d/2} (t/n^{-\theta})^{d/2} (c\alpha^d)^2 (\log n)^2 / n^2 . \quad (8)$$

Also

$$\begin{aligned} P[M_n(t_j) > r_n] P[M_n(t_k) > r_n] &= \{ (1 - P[R(t_j) > r_n]) (1 - P[R(t_k) > r_n]) \}^n \\ &\geq (1 - P[R(t_j) > r_n] - P[R(t_k) > r_n])^n . \end{aligned} \quad (9)$$

Combining (7)-(9) , we obtain the estimate

$$\begin{aligned}
 & \left[\frac{P[M_n(t_j) > r_n, M_n(t_k) > r_n]}{P[M_n(t_j) > r_n]P[M_n(t_k) > r_n]} \right] \\
 & \leq \left[\frac{1 - P[R(t_j) \leq r_n] - P[R(t_k) \leq r_n] + (t^2/(t-\delta))^{d/2} (c\alpha^d)^2 (\log n)^2 n^{\theta d/2-2}}{1 - P[R(t_j) \leq r_n] - P[R(t_k) \leq r_n]} \right]^n \\
 & = (1 + (t^2/(t-\delta))^{d/2} (c\alpha^d)^2 (\log n)^2 n^{\theta d/2-2} (1+o(1)))^n \\
 & \leq \exp(kn^{\theta d/2-1} (\log n)^2) \quad (\text{some constant } k) \\
 & \rightarrow 1 \quad (n \rightarrow \infty).
 \end{aligned} \tag{10}$$

Convergence in (10) is uniform over $\{(t_j, t_k) : t-\delta \leq t_j < t_k < t\}$.

Since $E(N_n) \rightarrow \infty$, by (5) and (6) we have

$$\begin{aligned}
 \limsup_{n \rightarrow \infty} (E(N_n^2))/(E(N_n))^2 & \leq \limsup_{n \rightarrow \infty} \left[\frac{2 \sum_{j < k} P[M_n(t_j) > r_n, M_n(t_k) > r_n]}{2 \sum_{j < k} P[M_n(t_j) > r_n]P[M_n(t_k) > r_n]} \right] \\
 & \leq 1 , \quad \text{by (10) .}
 \end{aligned}$$

Hence, recalling the Cauchy-Schwarz estimate (3) ,

$$\begin{aligned}
 P[M_n^*(t-\delta, t) > r_n] & \geq P(N_n > 0) \geq (E(N_n))^2 / E(N_n^2) \\
 & \rightarrow 1 \quad \text{as } n \rightarrow \infty .
 \end{aligned}$$

The proof of (***) is complete, and the result follows. \square

8.19 The minimal function of M_n . We now consider the random

variable $(M_n)_*$ given by the minimum over an interval of the process M_n , i.e.

$$(M_n)_*(t_1, t_2) = \inf\{M_n(t) : t_1 \leq t \leq t_2\}$$

where $0 < t_1 < t_2 \leq \infty$. We shall derive the limiting distribution of

a suitable re-normalisation of $(M_n)_*(t_1, t_2)$. Set

$$R_*(t_1, t_2) = \inf\{R(t) : t_1 \leq t \leq t_2\}.$$

Then $(M_n)_*(t_1, t_2)$ is the minimum of n independent copies of the random variable $R_*(t_1, t_2)$. Hence by standard extreme value theory we shall be able to describe the limiting distribution of $(M_n)_*(t_1, t_2)$ (suitably re-normalised) if we can describe $P[R_*(t_1, t_2) \leq r]$ as $r \downarrow 0$.

If $d = 1$, then $P[R_*(t_1, t_2) = 0] > 0$. Hence

$$P[(M_n)_*(t_1, t_2) > 0] \rightarrow 0 \quad (n \rightarrow \infty).$$

From now on we shall consider only $d > 2$. Let π_d be the volume of the unit ball in \mathbb{R}^d , and let $\sigma_d = d\pi_d$; $R(t)$ has the probability density function

$$f_t(x) = \sigma_d (2\pi t)^{-d/2} e^{-x^2/2t} x^{d-1} \quad (x \geq 0).$$

For $0 < s_1 < s_2 \leq \infty$, and $r > 0$, let

$$Q(s_1, s_2) = P[R_*(t_1, t_2) \leq r].$$

8.20 LEMMA. Suppose $d > 2$. If $0 < t_1 < t_2 \leq \infty$, then

$$Q(t_1, t_2, R) = \sigma_d (2\pi)^{-d/2} (t_1^{1-d/2} - t_2^{1-d/2}) r^{d-2} (1+o(1)) \quad (r \downarrow 0).$$

Proof. First we consider the case $t_2 = \infty$. For $0 < t < \infty$,

$$Q(t, \infty, R) = P[R(t) \leq r] +$$

$$+ \sigma_d (2\pi t)^{-d/2} \int_{x=r}^{\infty} P[R_*(t_1, \infty) \leq r | R(t_1) = x] e^{-x^2/2t} x^{d-1} dx.$$

By standard probabilistic potential theory, the conditional probability inside the integral is a function $(g(x), r \leq x < \infty)$

satisfying

$$g''(x) + ((d-1)/x) g'(x) = 0$$

subject to the boundary conditions $g(r) = 1$ and $\lim_{x \rightarrow \infty} g(x) = 0$.

The solution is

$$g(x) = x^{2-d} r^{d-2}, \quad r \leq x < \infty.$$

Hence

$$\begin{aligned} Q(t, \infty, r) &= O(r^d) + \sigma_d (2\pi t)^{-d/2} r^{d-2} \int_r^\infty x \exp^{-x^2/2t} dx \\ &= \sigma_d (2\pi)^{-d/2} t^{1-d/2} r^{d-2} (1+o(1)) \quad (r \downarrow 0) \end{aligned} \quad (1)$$

as desired. For $0 < t_1 < t_2 < \infty$, we estimate $Q(t_1, t_2, r)$ by

$$\begin{aligned} Q(t_1, \infty, r) - Q(t_2, \infty, r) &\leq Q(t_1, t_2, r) \\ &\leq Q(t_1, \infty, r) - Q(t_2, \infty, r) + P[\{R_*(t_1, t_2) \leq r\} \cap \{R_*(t_2, \infty) \leq r\}]. \end{aligned}$$

To prove the desired result, it remains to prove

$$P(A) = o(r^{d-2}) \quad (r \downarrow 0)$$

where $A = \{R_*(t_1, t_2) \leq r\} \cap \{R_*(t_2, \infty) \leq r\}$.

Set

$$T = \inf\{t \geq t_1 : R(t) \leq r\}.$$

We divide the above event A into $A \cap \{T < t_2 - r^{2-\epsilon}\}$ and $A \cap \{T \geq t_2 - r^{2-\epsilon}\}$, where $\epsilon > 0$. By the strong Markov property of the Bessel process, and the fact that $Q(t, \infty, r)$ is monotone decreasing in t ,

$$\begin{aligned} P[\{T < t_2 - r^{2-\epsilon}\} \cap A] &= P[\{T < t_2 - r^{2-\epsilon}\} \cap \{R_*(t_2, \infty) \leq r\}] \\ &\leq Q(t_1, \infty, r) Q(r^{2-\epsilon}, \infty, 2r). \end{aligned}$$

Hence, by the Brownian scaling property,

$$\begin{aligned} P[\{T < t_2 - r^{2-\epsilon}\} \cap A] &\leq Q(t_1, \infty, r) Q(1, \infty, 2r(r^{-1+\epsilon/2})) \\ &= o(r^{d-2}) \quad (r \downarrow 0). \end{aligned}$$

Moreover, for $0 < r < 1$,

$$\begin{aligned} P[A \cap \{T \geq t_2 - r^{2-\epsilon}\}] &\leq Q(t_2 - r^{2-\epsilon}, t_2, r) \\ &\leq P[R(t_2 - r^{2-\epsilon}) < 2r^{1-\epsilon}] \\ &+ P[\sup\{R(t_2 - r^{2-\epsilon} + t) - R(t_2 - r^{2-\epsilon}) : 0 \leq t \leq r^{2-\epsilon}\} \geq r^{1-\epsilon}] \\ &= O(r^{d(1-\epsilon)}) + P[R^*(1) > r^{-\epsilon/2}] \\ &= o(r^{d-2}) \quad (r \downarrow 0) \end{aligned}$$

provided that $\epsilon < 2/d$.

Hence $P(A) = o(r^{d-2})$ as $r \downarrow 0$, as required. \square

8.21 THEOREM. if $d > 2$, and $0 < t_1 < t_2 \leq \infty$, then the random variables

$$n^{1/(d-2)} (M_n)_*(t_1, t_2)$$

converge in law to a random variable with distribution function

$$F(y) = 1 - \exp\{-\sigma_d(2\pi)^{-d/2} (t_1^{1-d/2} - t_2^{1-d/2}) y^{d-2}\}, \quad y \geq 0.$$

Proof. Since $(M_n)_*$ is the minimum of n independent copies of $R_*(t_1, t_2)$, the result follows easily from Lemma 8.20 (cf. Galambos (1978), corollary 1.3.2).

8.22 Remarks. By similar methods (see also the proof of Proposition 1 of Penrose (1988b)), given any finite collection of intervals $\{[a_j, b_j], 1 \leq j \leq J\}$ in $(0, \infty)$, we can find the limiting joint

distribution of the random variables $n^{1/(d-2)} (M_n)_*(a_j, b_j)$, as $n \rightarrow \infty$.

We can now contrast the rates of convergence to zero, as $n \rightarrow \infty$, of the distributions of the random variables $M_n(t)$, $M_n^*(s, t)$ and $(M_n)_*(s, t)$ for fixed s and t . Here $M_n^*(s, t)$ is defined to be the maximum of $M_n(\cdot)$ over the interval $[s, t]$. By "rate of convergence" of a sequence of probability distributions of random variables X_n to zero, we mean the rate of convergence to zero of a sequence a_n such that $a_n^{-1} X_n$ converge in law to a finite positive random variable.

By the one-dimensional case of Proposition 8.4, the law of $M_n(t)$ converges to zero in proportion to $n^{-1/d}$.

By Theorem 8.18, the law of $M_n^*(s, t)$ converges to zero in proportion to $(n/\log n)^{-1/d}$.

By Theorem 8.21, the law of $(M_n)_*(s, t)$ converges to zero in proportion to $n^{-1/(d-2)}$ ($d > 2$).

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